Minor kernelization of embedded graphs

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LaBRI

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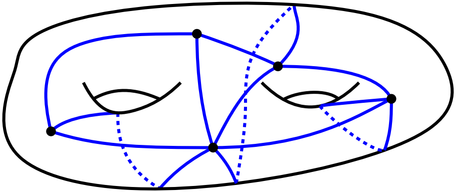
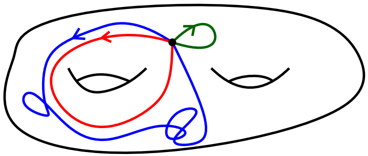
Combinatorial surface

A **combinatorial surface** is a graph $G = (V, E)$ cellularly embedded on a topological orientable compact surface S of genus g.
n is the number of vertices, e the number of edges and f the number of faces.
$n - e + f = 2 - 2g$.



Homotopy

Two closed loop γ and γ' on a surface are homotopic if one can be continuously deformed into the other.



- $\pi_1(S, v)$: group of loops based at v under homotopy
- $\mathcal{L}S$: set of loops under free homotopy

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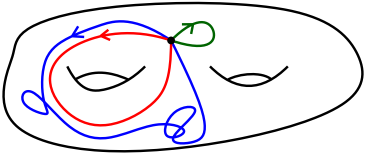

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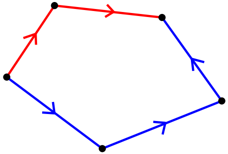
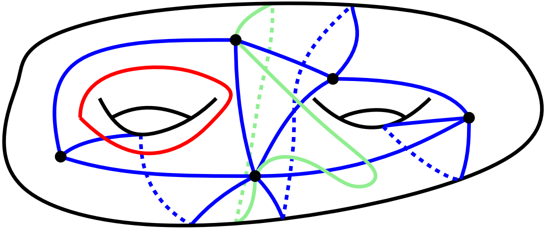
On a combinatorial surface, homotopy is the closure of the following relation on faces :



Length spectrum

The **length spectrum** is the function $\mu_G : \begin{array}{ccc} \mathcal{L}S & \to & \mathbb{N} \\ \gamma & \mapsto & \inf_{\gamma' \sim \gamma} cr(\gamma, G) \end{array}$



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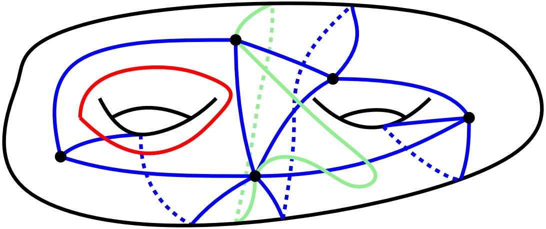

If H is a minor of G then $\mu_H \leqslant \mu_G$.
A **kernel** is an embedded graph G such that all proper minors H of G satisfy $\mu_H < \mu_G$.

Computing μ_G

c closed curve of length ℓ

Theorem (Colin de Verdière and Erickson, 2010)

Assume $g \geqslant 2$.
After $O(gn\log(gn))$ time preprocessing, $\mu_G([c])$ can be computed in time $O(gn\ell\log(n\ell))$.

Theorem (Delecroix, Ebbens, Lazarus, Yakovlev, 2024)

Assume $g = 1$.
After $O(n^2 \log\log n)$ time preprocessing, $\mu_G([c])$ can be computed in time $O(\ell + \log n)$.

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Theorem (Despré, Lazarus 2019 and Dubois 2024)

Assume G is a kernel.
$\mu_G([c])$ can be computed in time $O(g(n+\ell)\log(n+\ell))$.

Our result

H is a *minor kernel* of G if H is a kernel and a minor of G and $\mu_H = \mu_G$

Theorem (Delecroix, F., Lazarus)

Given a graph G of genus $g \geqslant 2$, a minor kernel can be computed in $O(n^3 \log n)$.

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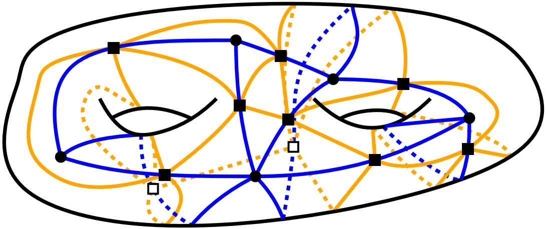
Medial graph

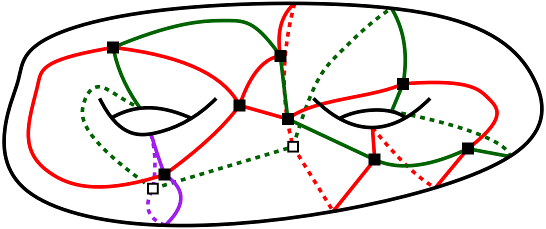
The medial graph M of G is defined by :

- one vertex on the middle of each edges
- one edge between the middles of two consecutive edges around each vertex



System of curves

The medial graph can be interpreted as a system of curves whose vertices are the transverse intersections of those curves.



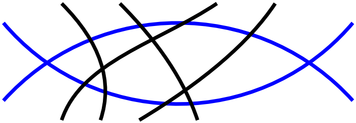
Universal covering

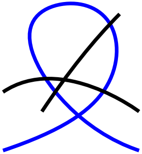
The **universal covering** $(\tilde{M}, p)$ of M on S is a subdivision of the plane and a projection such that

- each face, edge or vertex of $\tilde{M}$ projects on a face, edge or vertex of M with same degree
- two adjacent faces of $\tilde{M}$ project on two adjacent faces of M
- two adjacent vertices of $\tilde{M}$ project on two adjacent vertices

Bigons and monogons

A **bigon** is a disk bounded by two lifts of curves in $\tilde{M}$.
A **monogon** is a disk bounded by a lift of curves in $\tilde{M}$.





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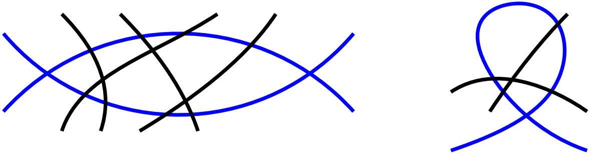

An **empty monogon** is a monogon without incident edges.
A **minimal bigon** is a bigon with no bigon nor monogon inside it.

Minimality

Theorem (Schrijver 1992)

G is not a kernel if and only if M contains either a minimal bigon or an empty monogon.

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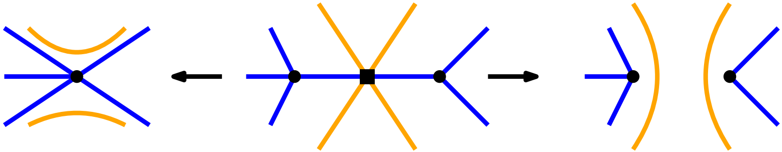
Moreover in that case a corner of such a bigon or monogon can be smoothed without changing the length spectrum.

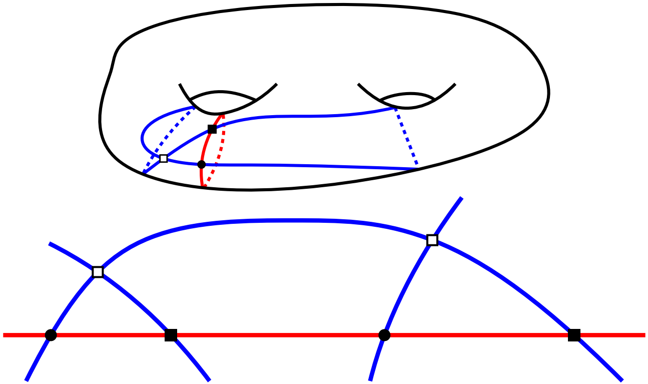


Length of a minimal bigon

Theorem (Delecroix, F., Lazarus)

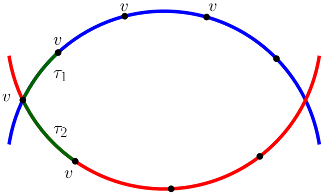
Assuming $g \geqslant 2$, the length of every minimal bigon in $\tilde{M}$ is at most $8n$.

Long bigon



Sketch of the proof

Goal : bound the number of lifts on one curve.



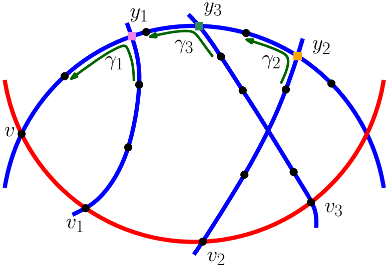
$$\pi_1(S, v) \simeq \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = e \right\rangle$$

Look at the subgroup of $\pi_1(S, v)$ generated by τ_1 and τ_2. It is isomorphic to one of the following groups :

- $\{1\}$
- $\mathbb{Z}$
- F_2

A ping pong lemma

Assume $< \tau_1, \tau_2 > \simeq F_2$.



Minimal bigon detection algorithm

- Find all bigon of length at most $8n$.
- Compute the area of all this bigon.
- Return the bigon with minimal area.

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Area computation: a Discrete Stokes formula for reduced graph and the tree/co-tree decomposition

Conclusion

Algorithm to compute a minor kernel of G:

1. Compute the medial graph M of G
2. Find an empty monogon or a minimal bigon in M
3. Smooth one of its corners and do the corresponding minor operation on G
4. Repeat this operation until there is no monogon nor bigon

This process compute a minor kernel in $O(n^3 \log(n))$.

Perspectives

- Length of any bigon ? Number of bigons ?
- Number of kernels with a given spectrum ?
- Given a spectrum, construct a graph with this spectrum.