An asymptotic rigidity property of chirotope extensions

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Geometric object $\rightsquigarrow$ Combinatorial object

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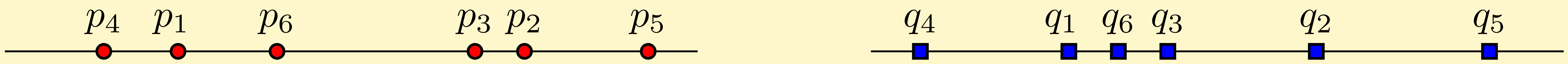

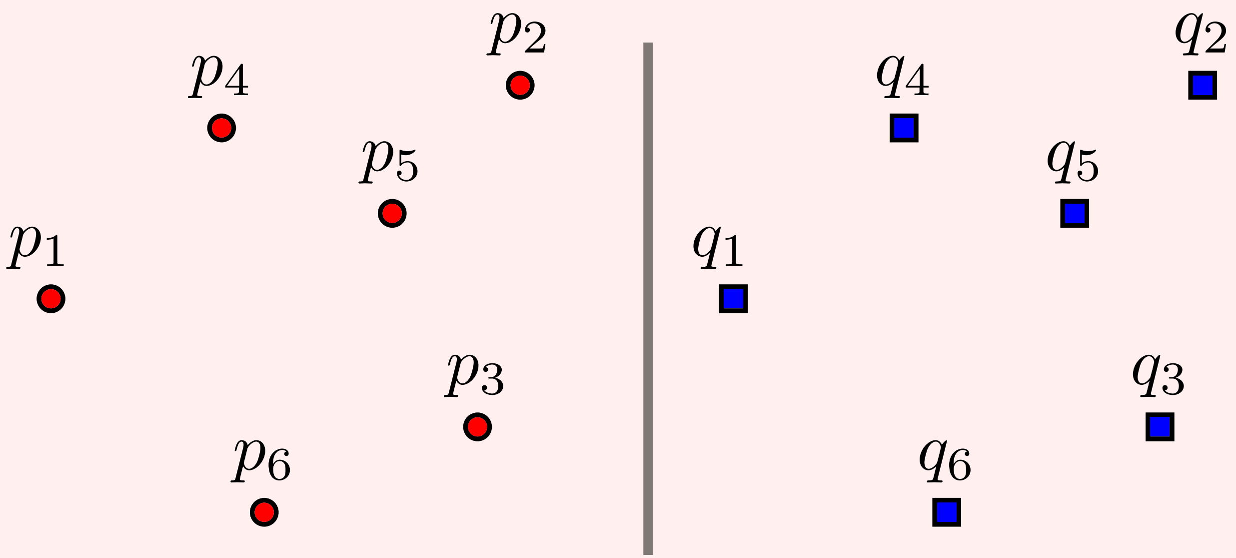

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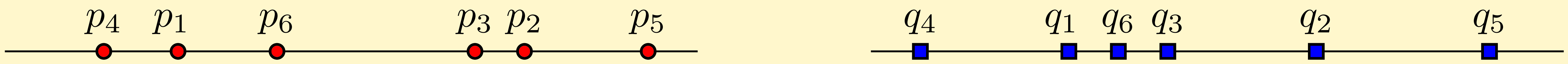

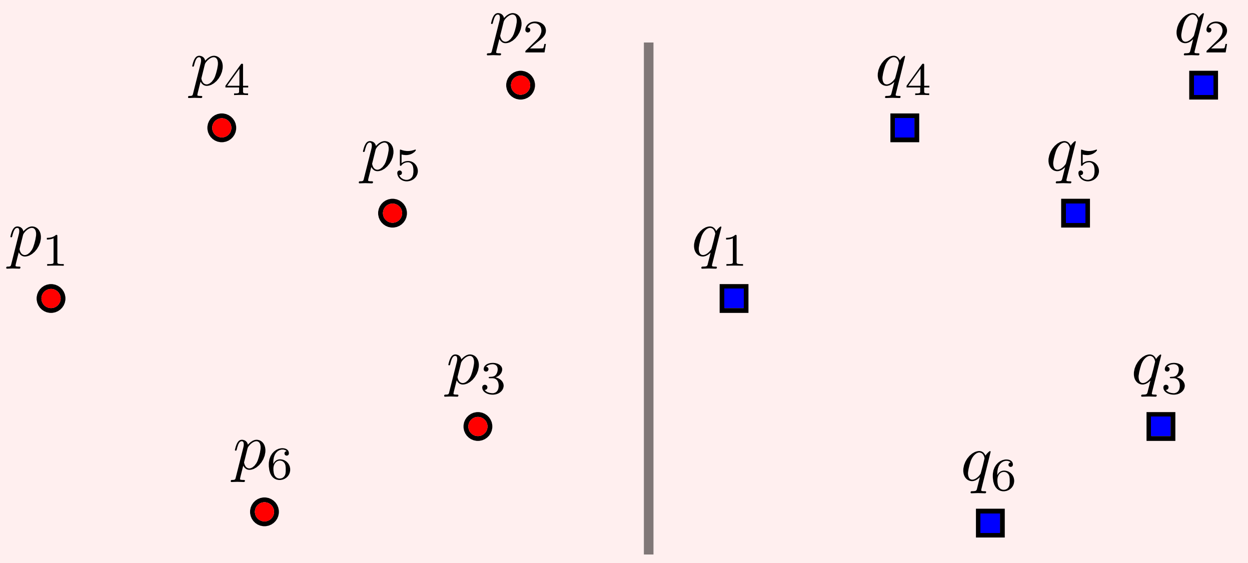

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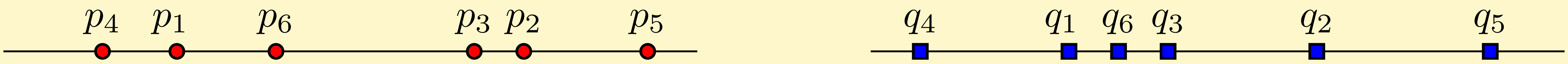

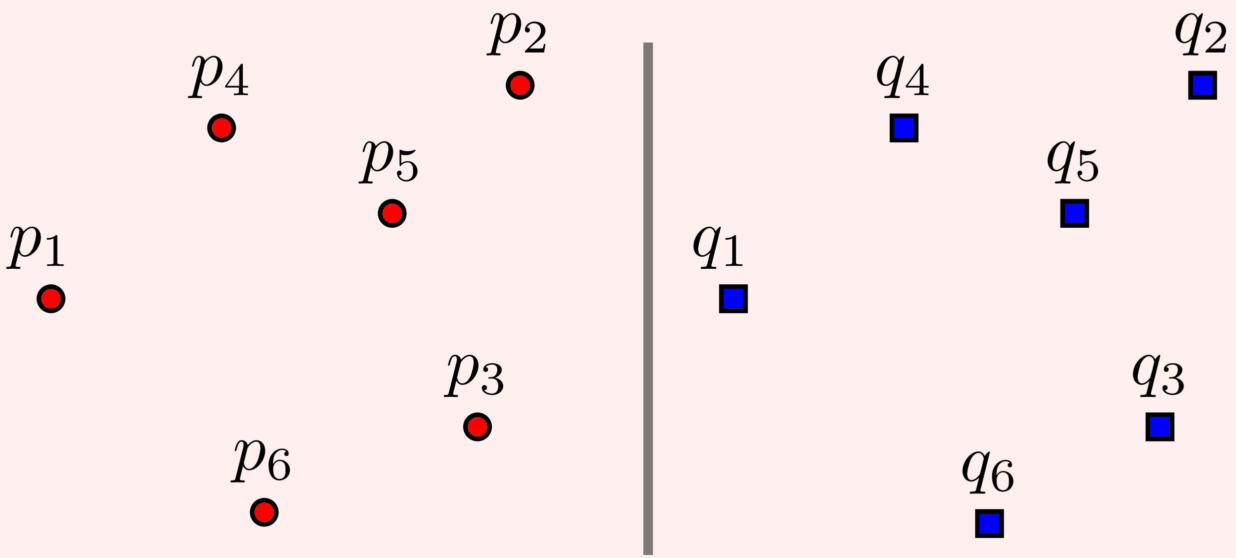

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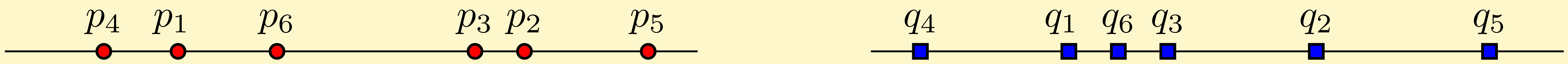

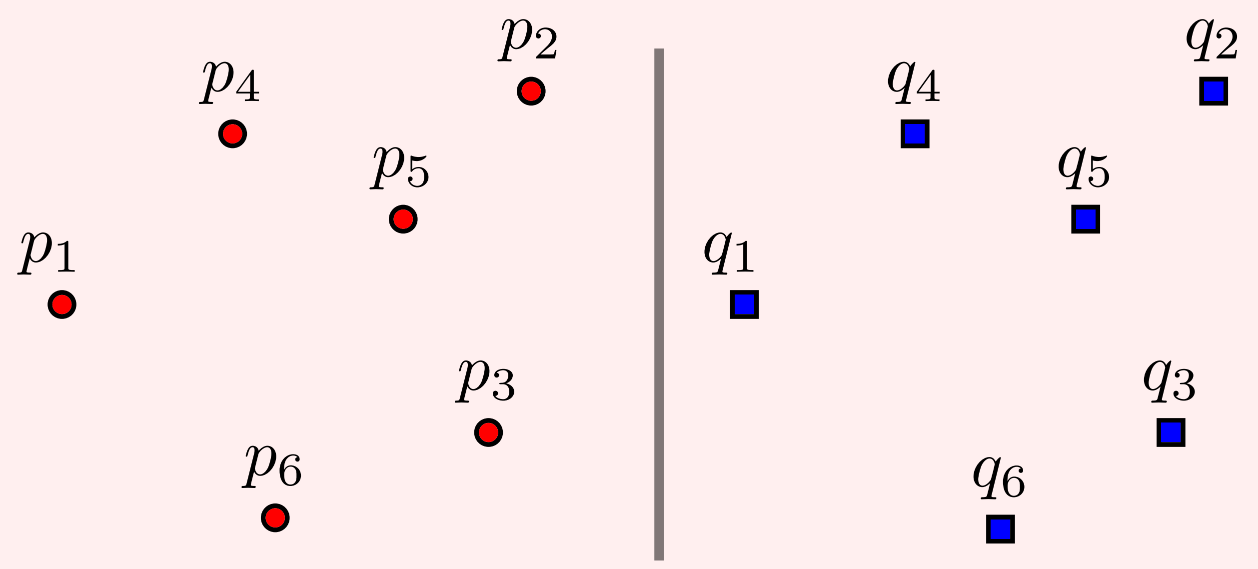

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Model many (geometric) structures.

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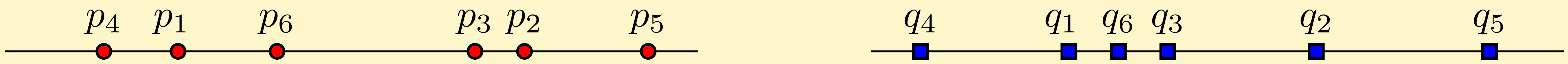

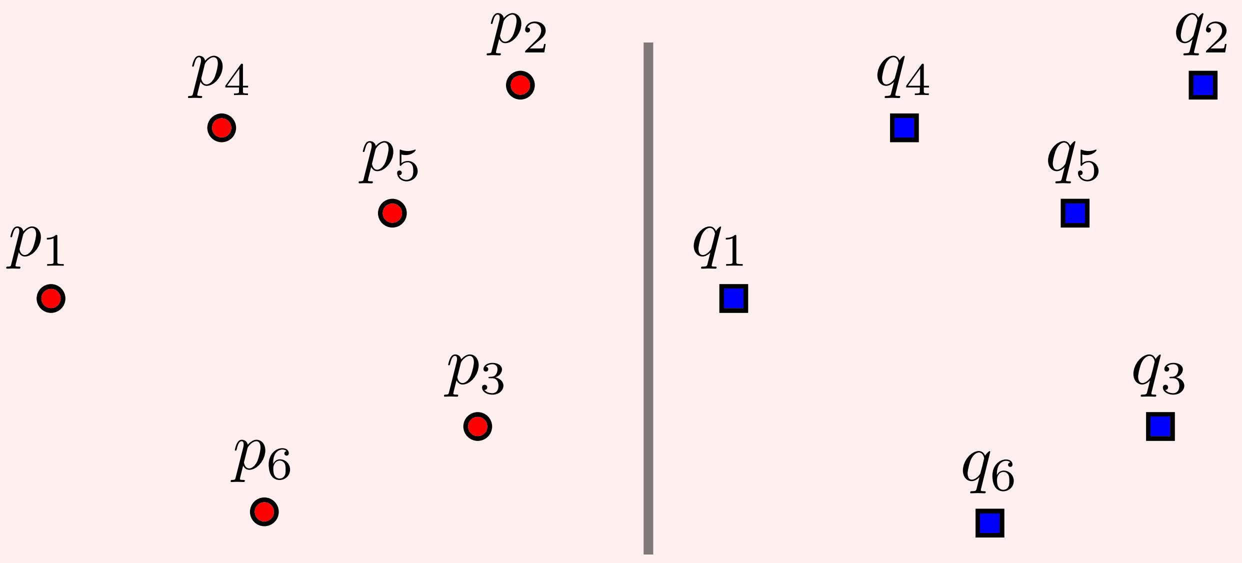

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Discovered and re-discovered.

[Bland 1974] [Las Vergnas 1975] [Folkman-Lauwrence 1978][Goodman-Pollack 1983] [Knuth 1992]

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$$R(f) = \{v \in \mathbb{R}^{(d+1)n} : \text{the chirotope of } (v_1, \ldots, v_{d+1}), (v_{d+2}, \ldots, v_{2d+2}), \ldots, (v_{(d+1)(n-1)+1}, v_{(d+1)n}) \text{ is } f\}$$

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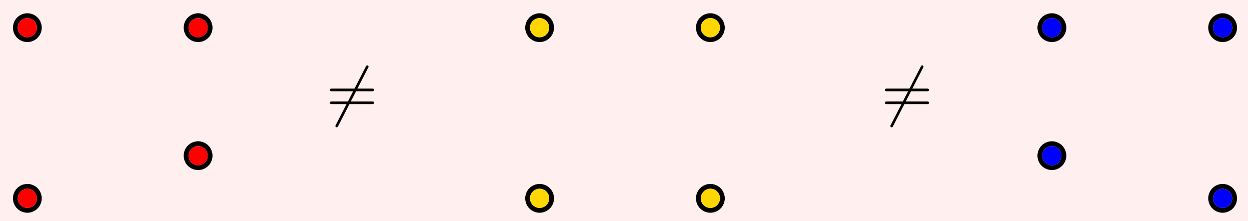

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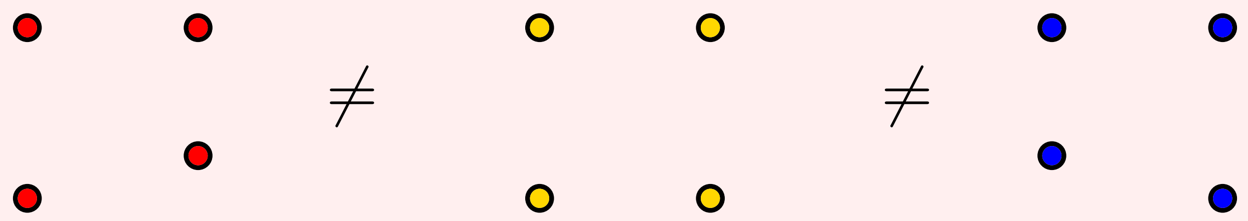
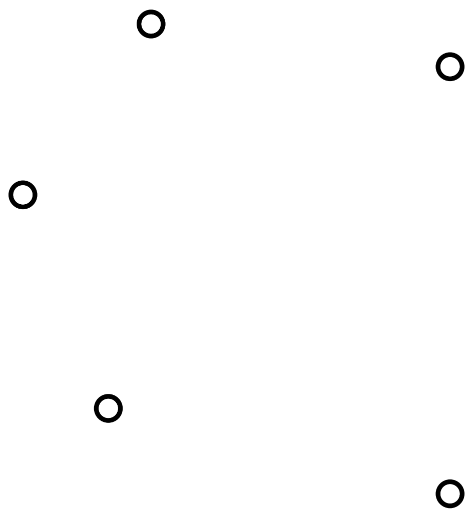
What if we add information ?

allowable sequences, CCC systems, sweep oriented matroids, adjoints of oriented matroids, strong geometries, ...



Realization spaces mess with extensions...

n points in convex position.



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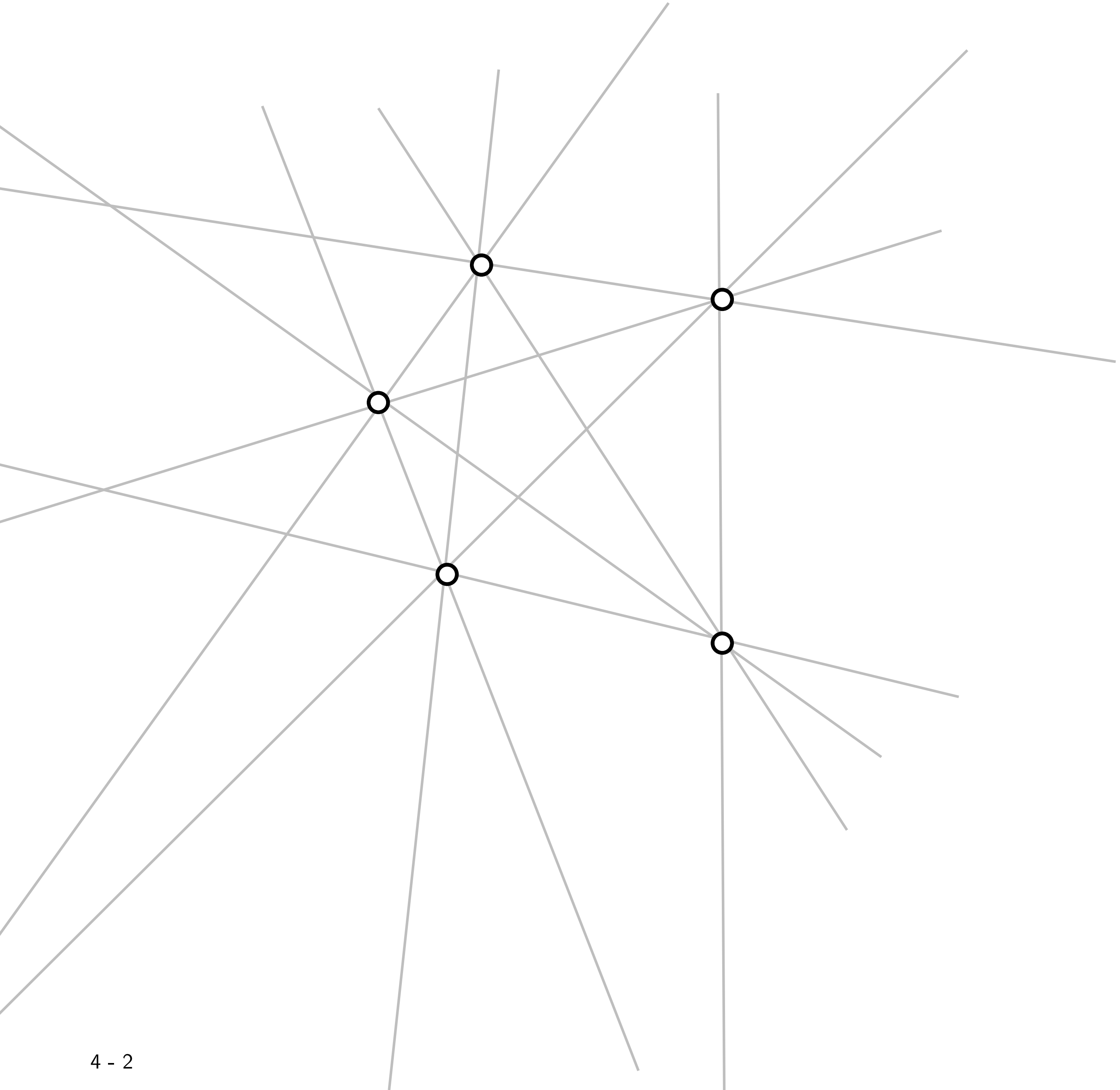

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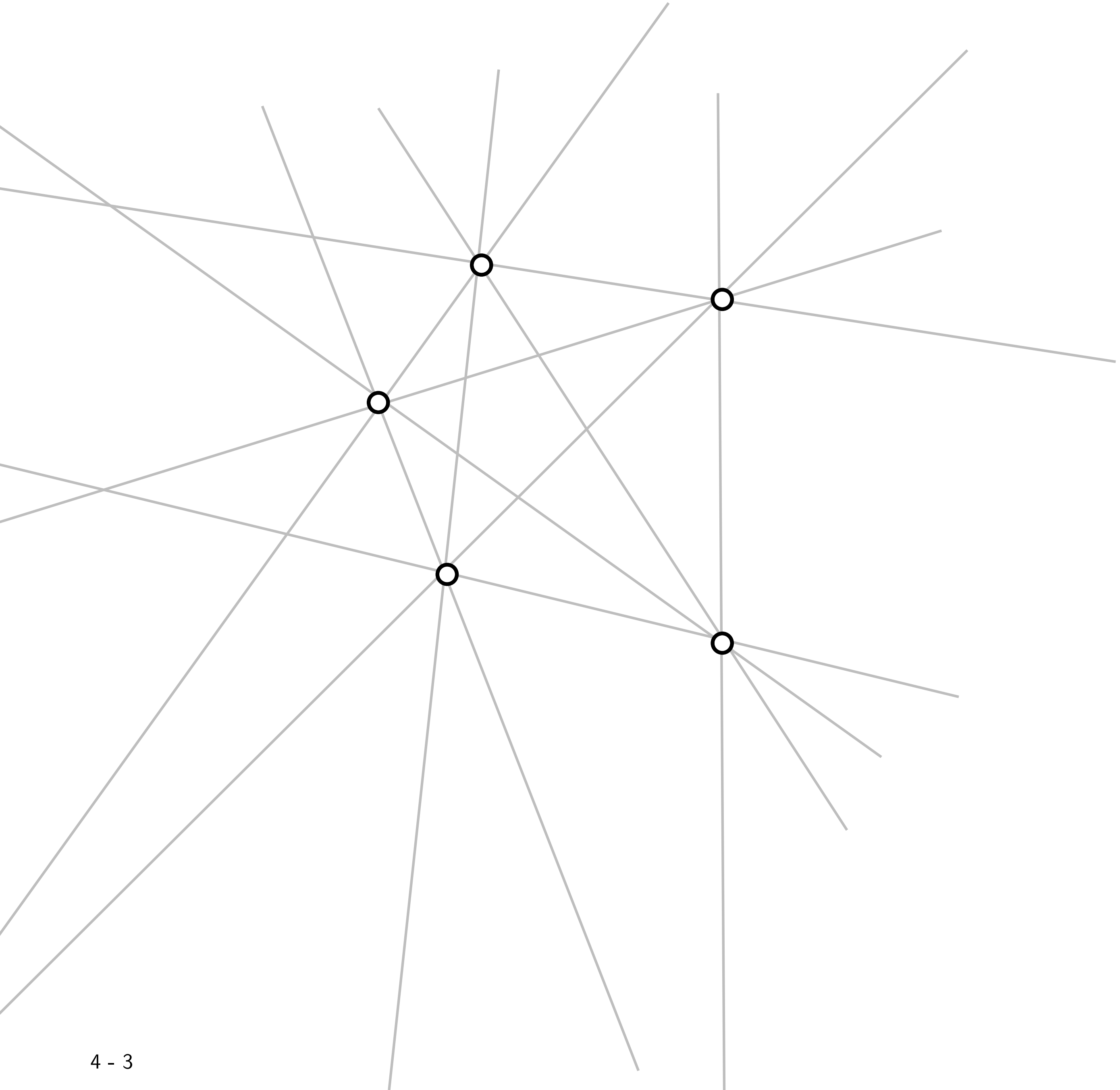

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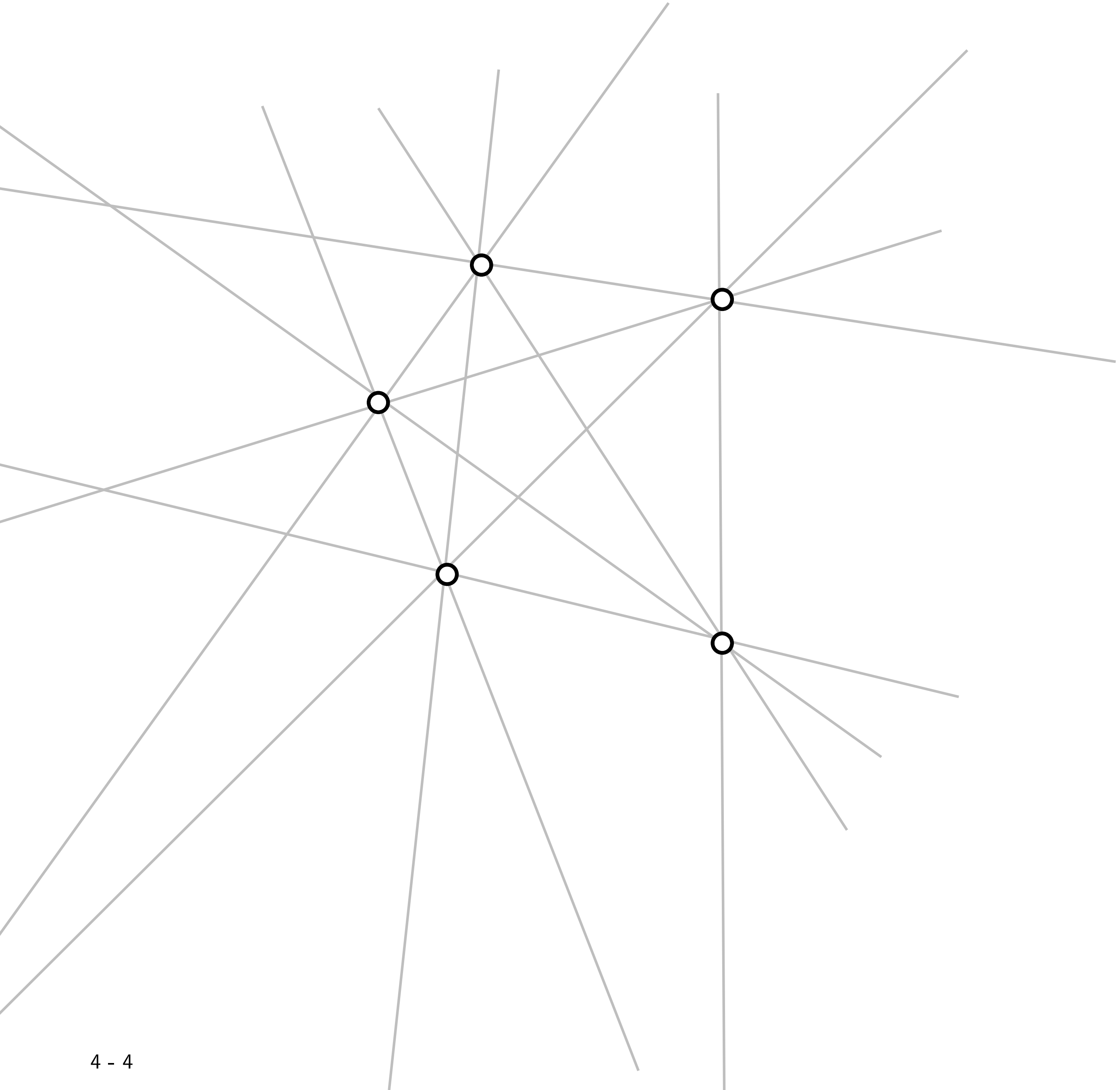

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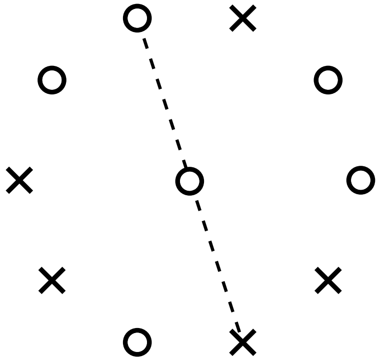

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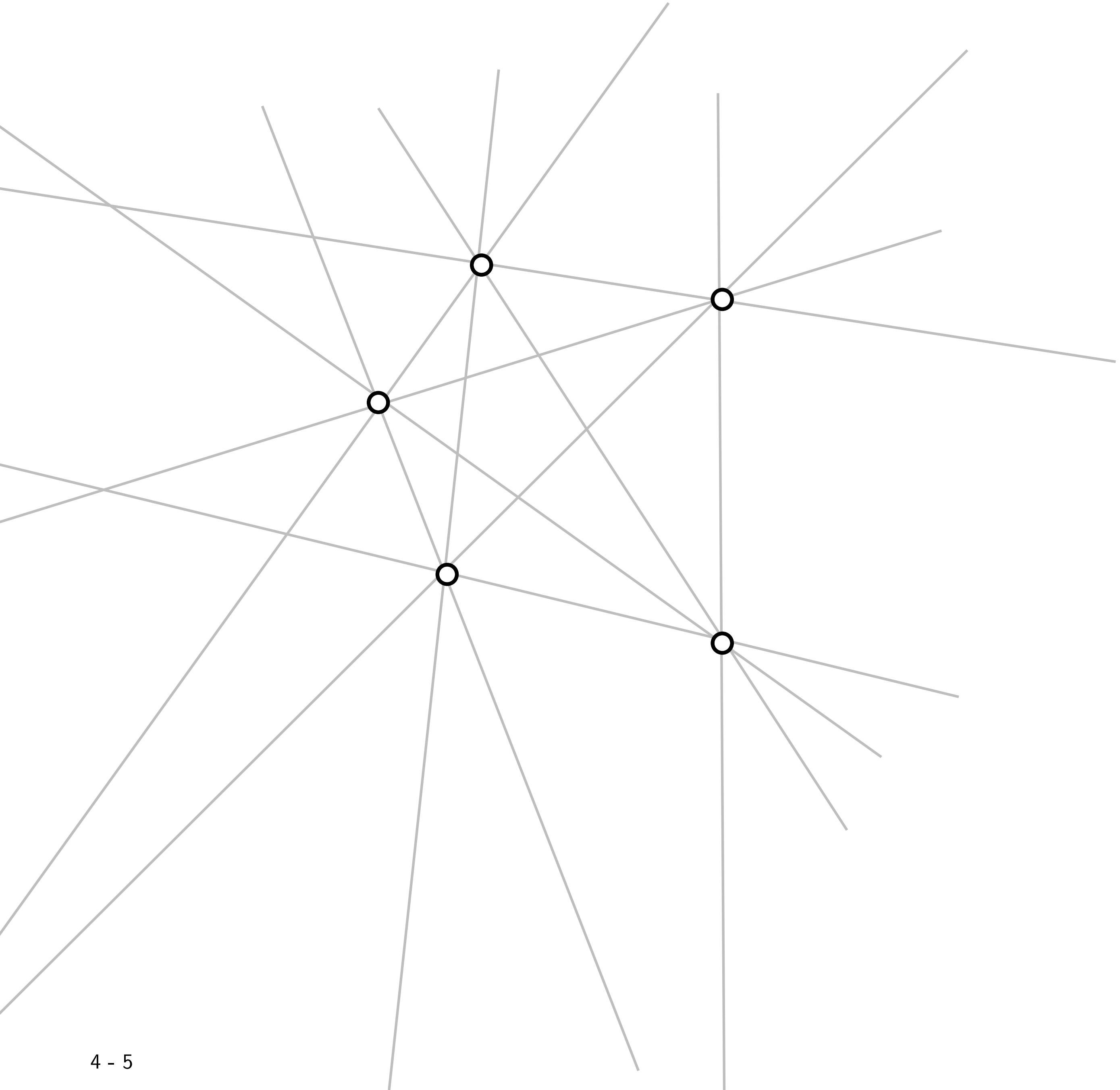

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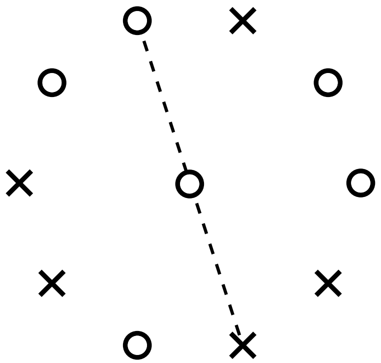

So we want to understand the
extension vs realization relation better...

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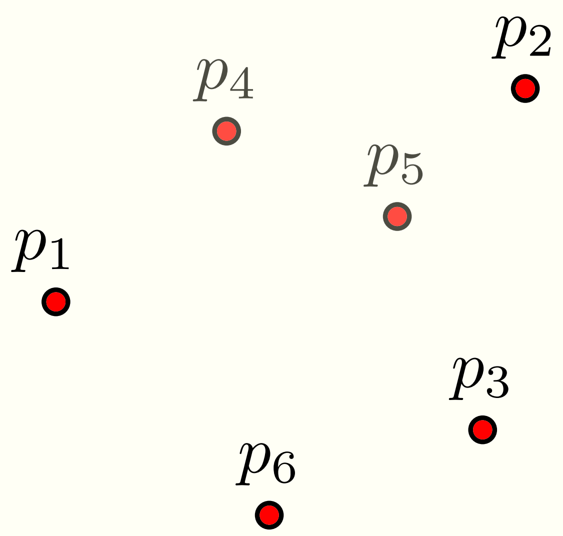

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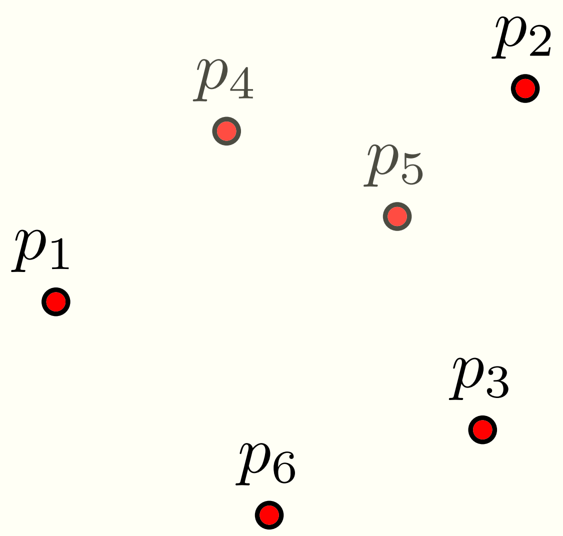

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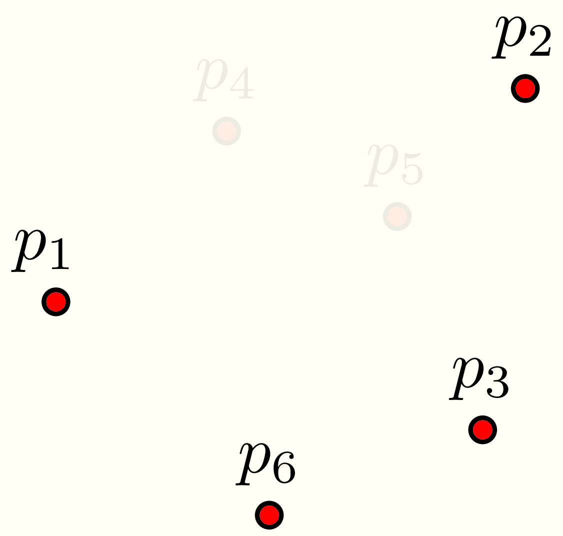

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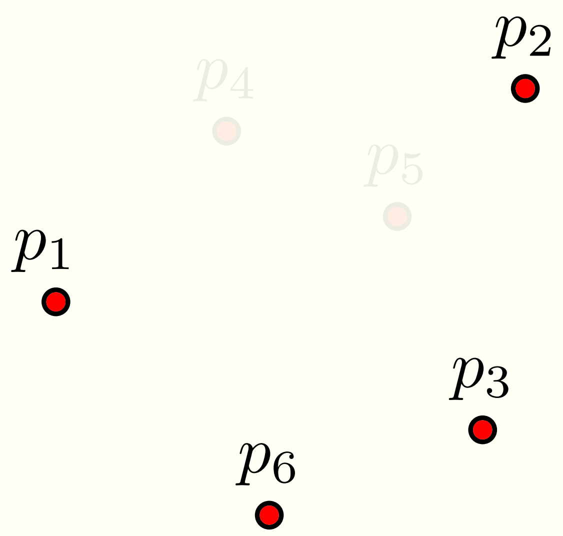

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A chirotope χ is **realizable on top of** $P \in (\mathbb{R}^d)^X$ if there exists an extension of P realizing χ.

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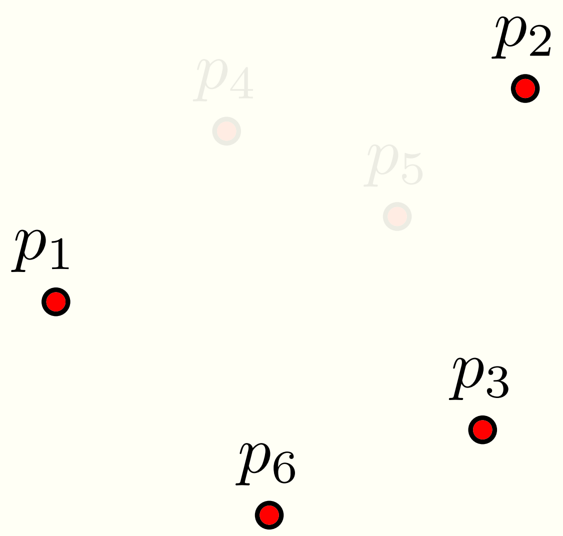

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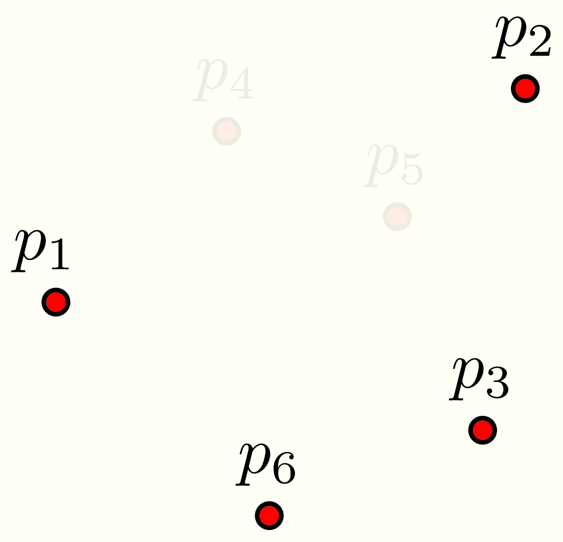

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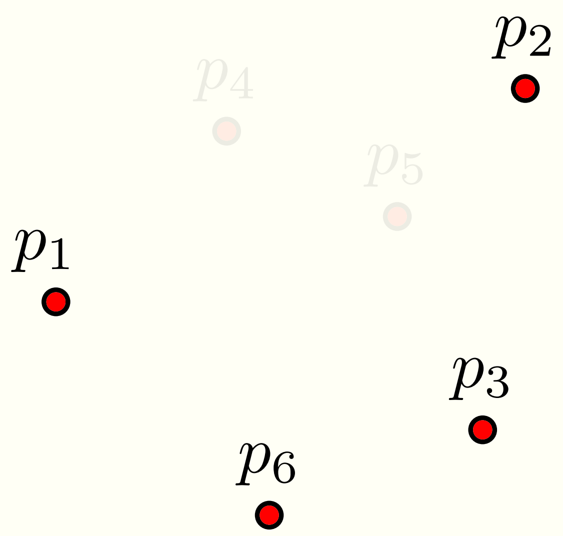

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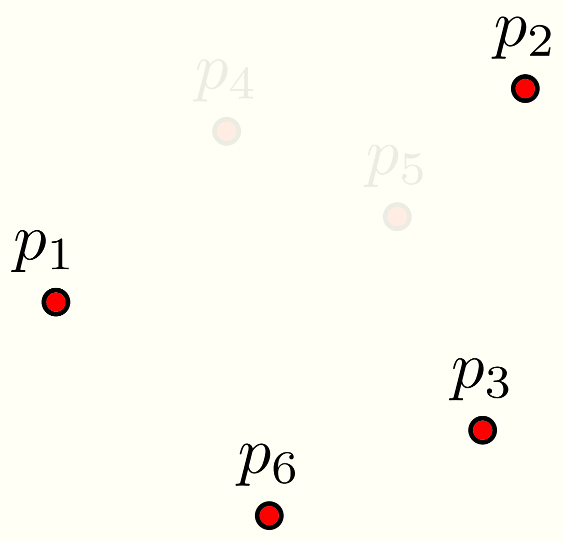

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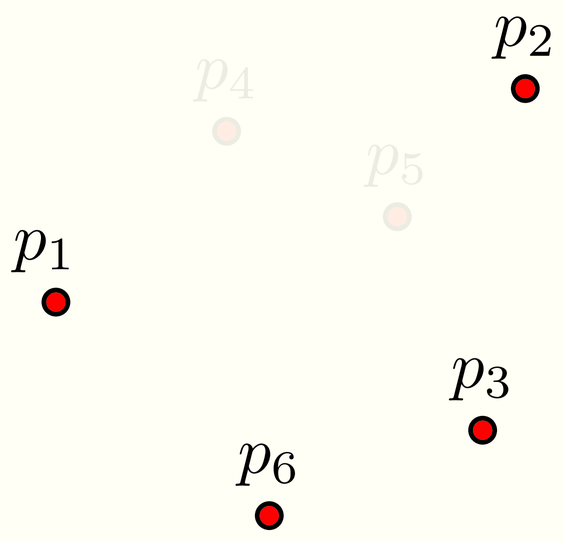

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$Y = \{1, 2, 3, 4, 5, 6\}$

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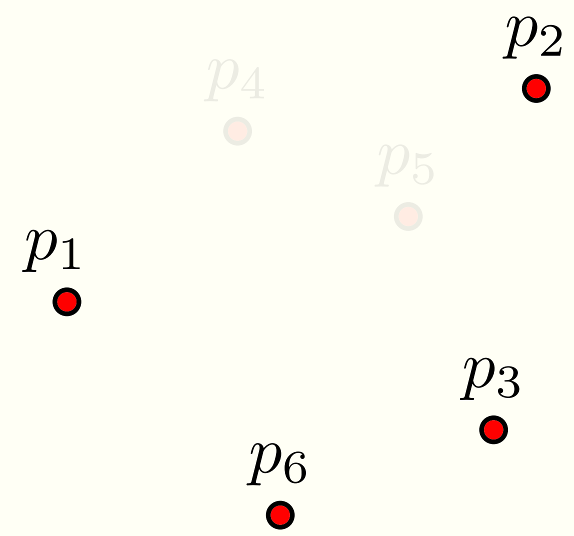
$X \subseteq Y$ are finite (label) sets and $P \in (\mathbb{R}^d)^Y$.

The **restriction** of $P \in (\mathbb{R}^d)^Y$ to X is $P_{|X} \stackrel{\text{def}}{=} \{p_i\}_{i \in X}$.

$P \in (\mathbb{R}^d)^Y$ **extends** $Q \in (\mathbb{R}^d)^X$ if $P_{|X} = Q$.

A chirotope χ is **realizable on top of** $P \in (\mathbb{R}^d)^X$ if there exists an extension of P realizing χ.



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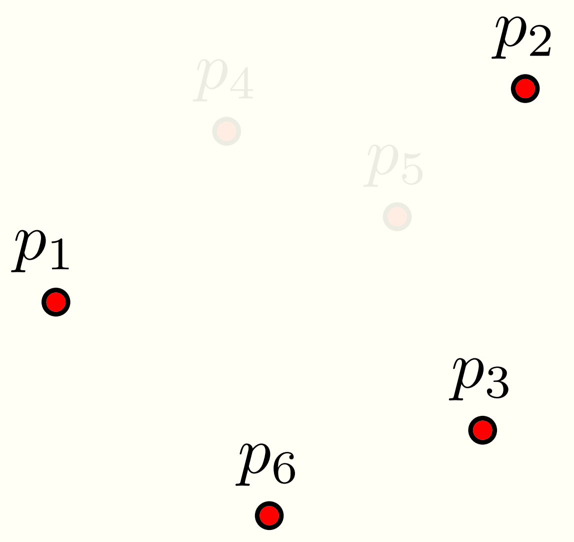

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New : $R_\infty(P) \overset{\text{def}}{=} \cap_{k \geq 0} R_k(P)$ is simple.

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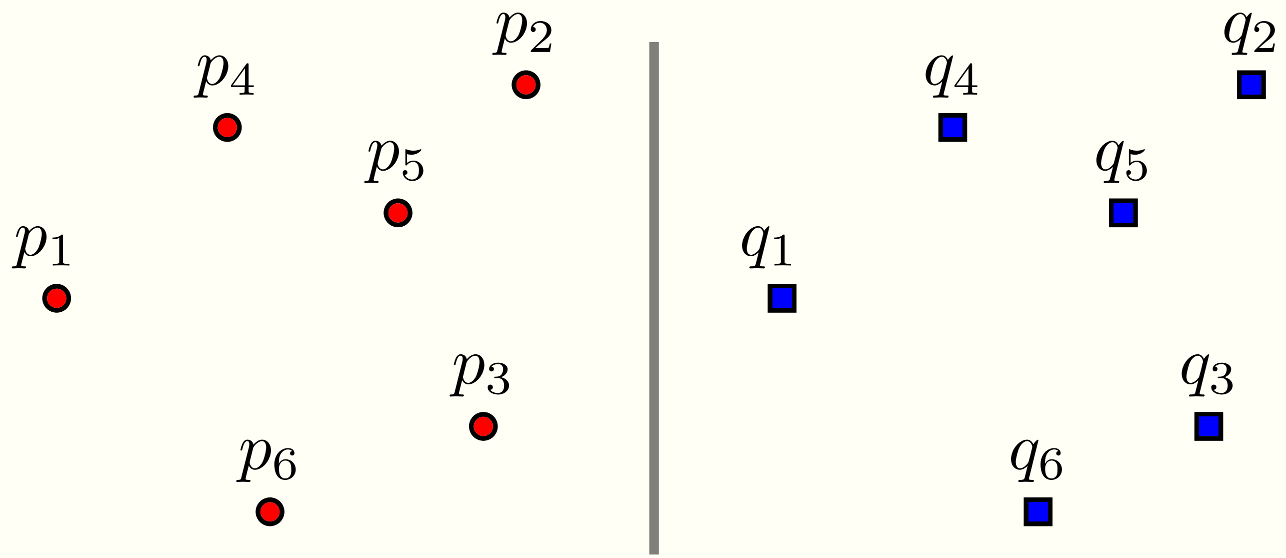
$\delta_C(P,Q)$ controls $\Delta_G(P,Q)$. A single extension of P suffices...

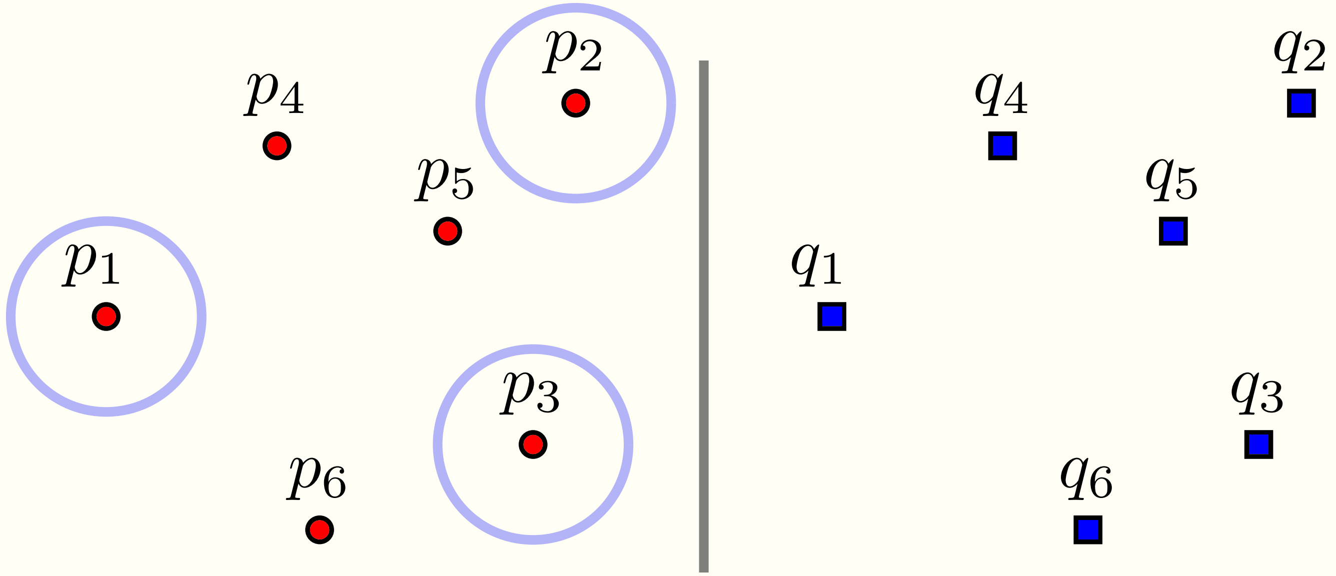
What's under the hood ?

Normalizing the problem ($d = 2$)



Normalizing the problem ($d = 2$)

▷ choose 3 points spanning $\mathbb{R}^2$



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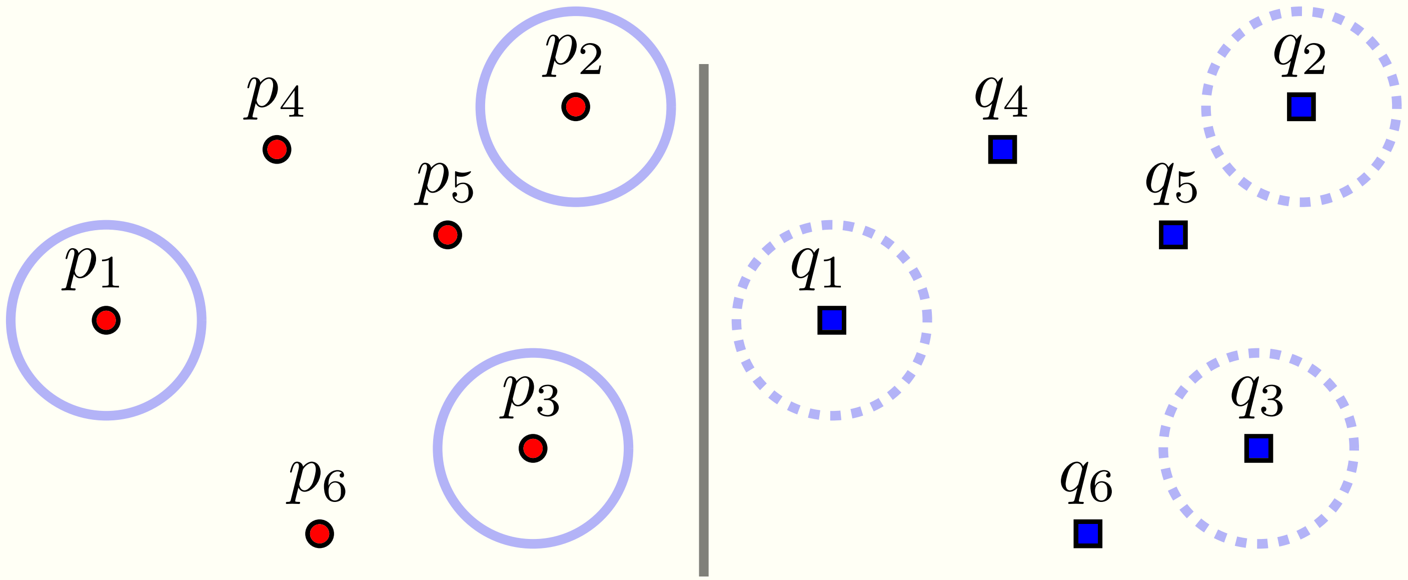

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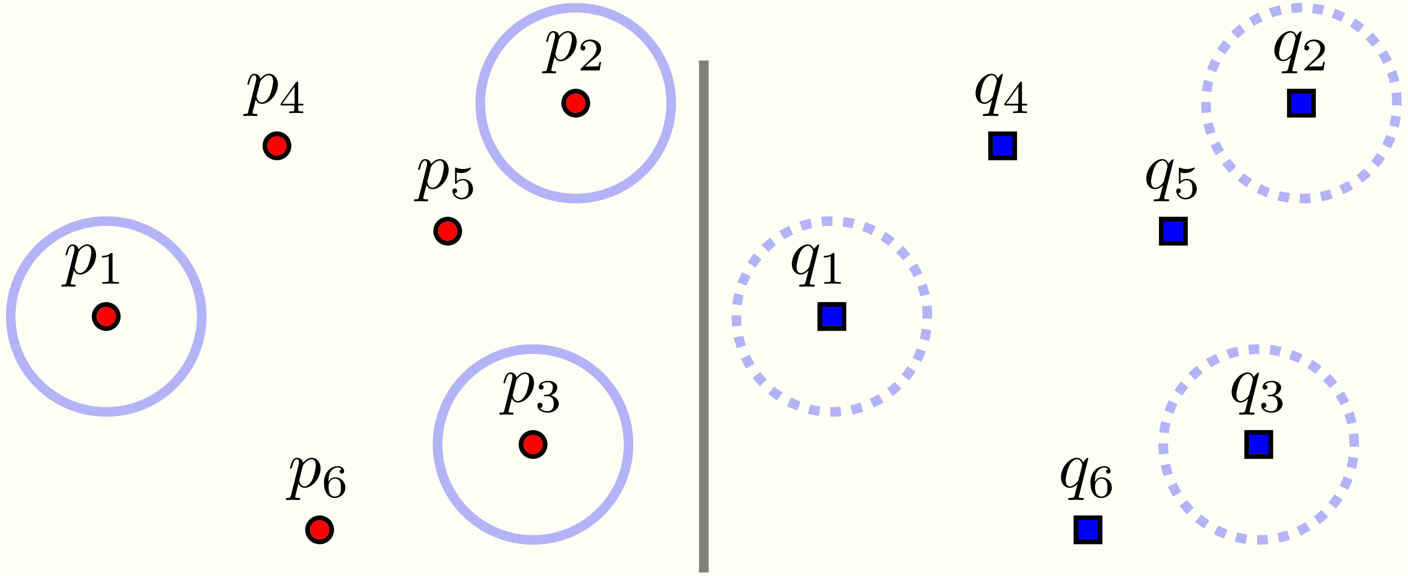

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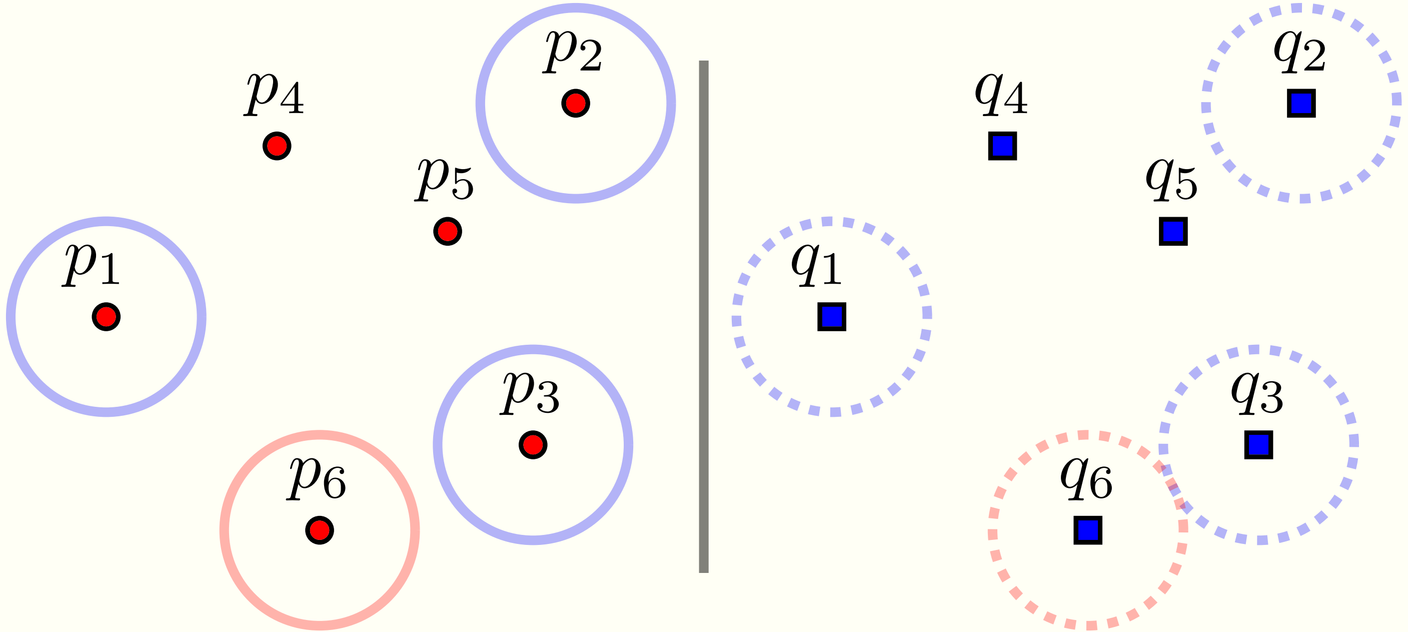

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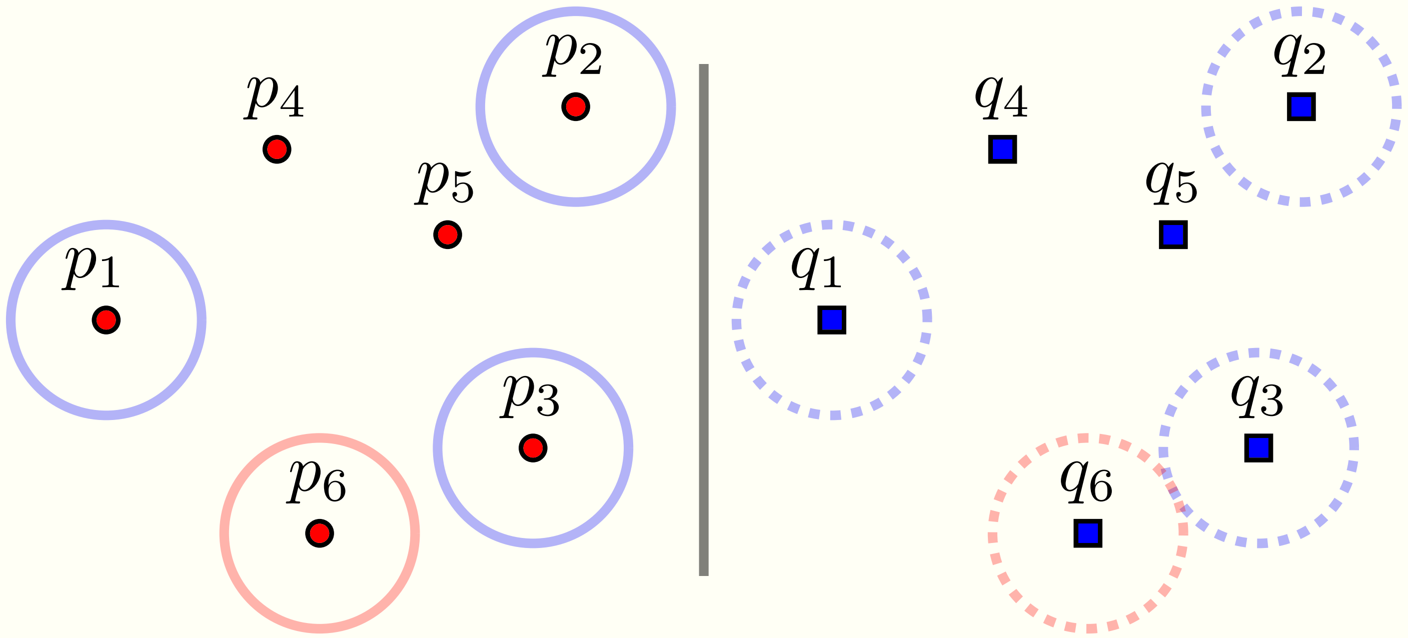

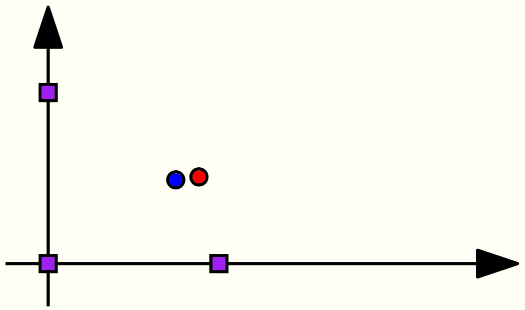

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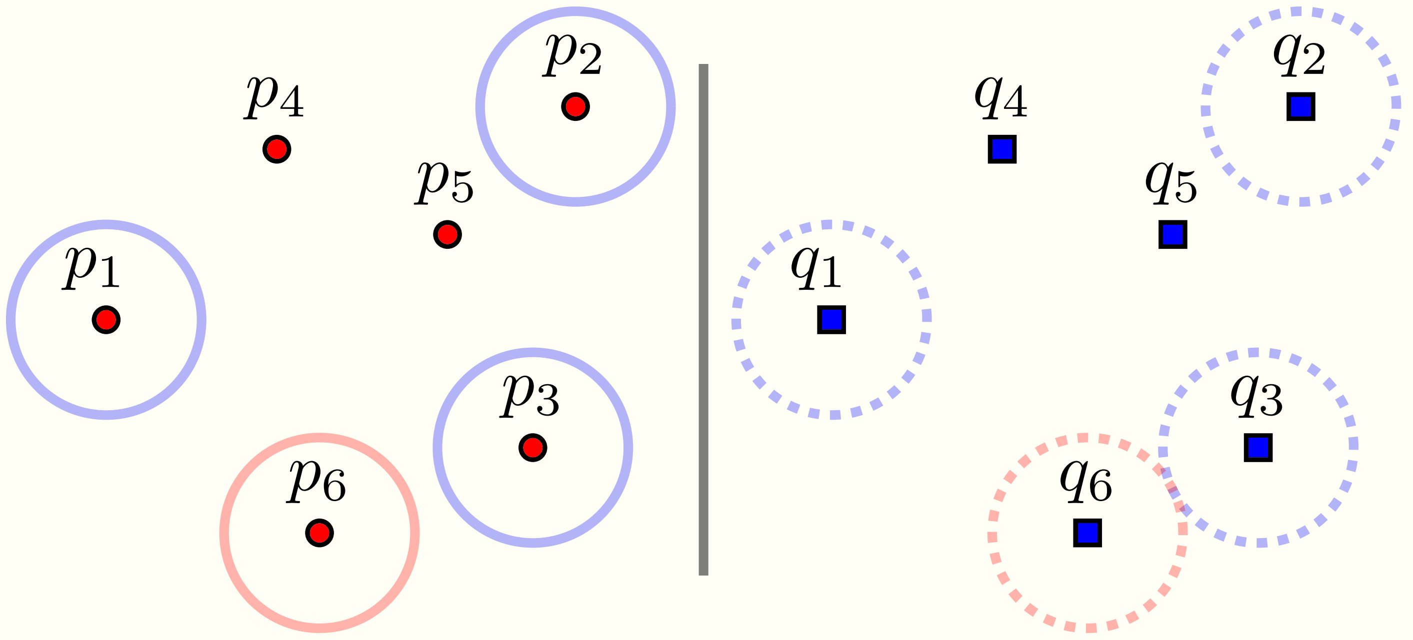
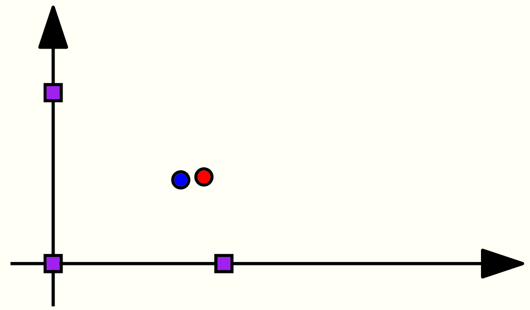
- ▷ use projections to make the problem almost 1d





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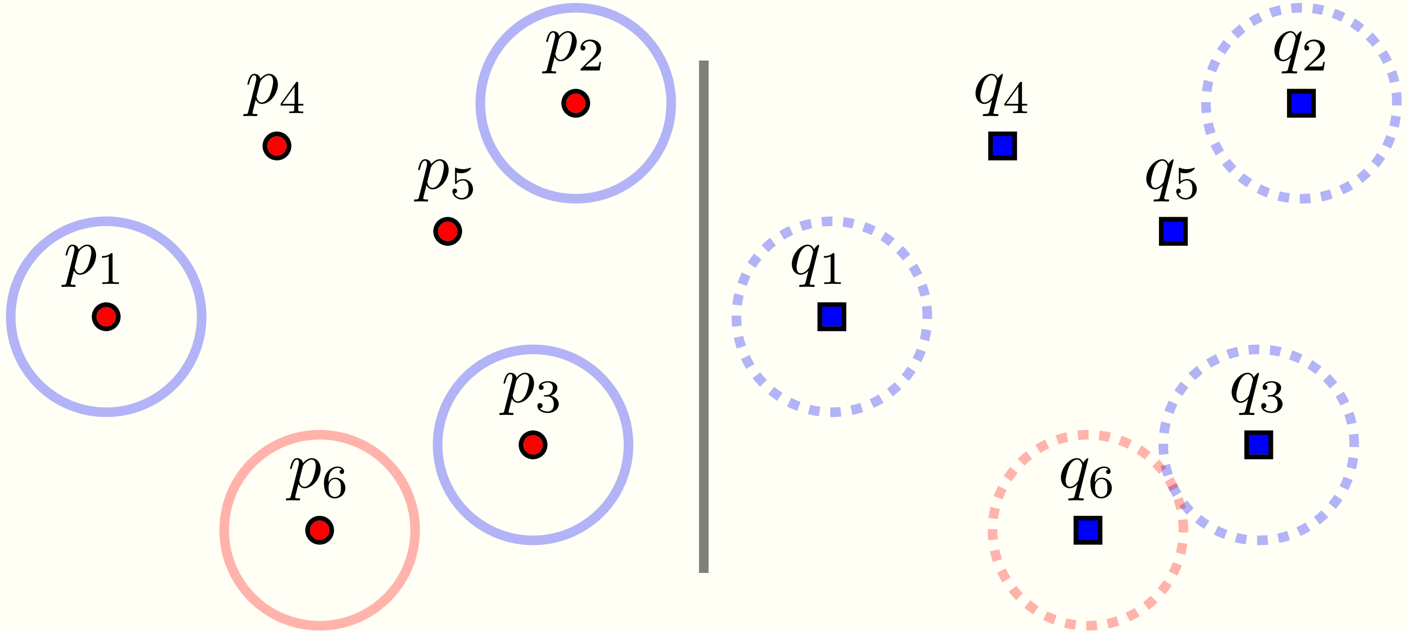

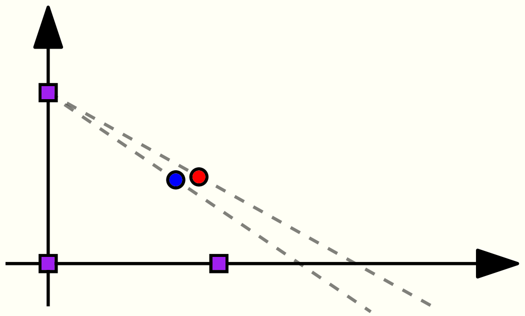

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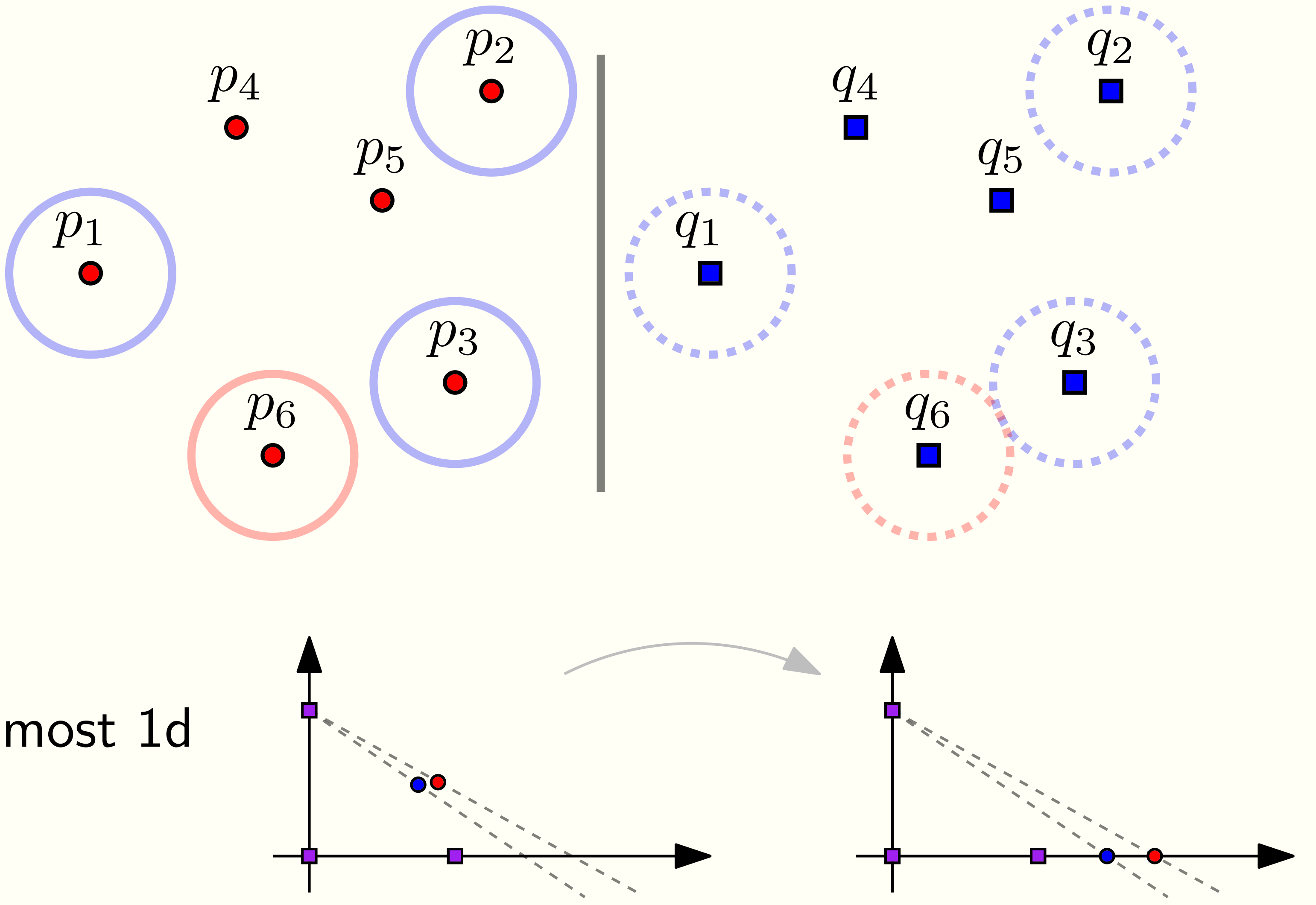

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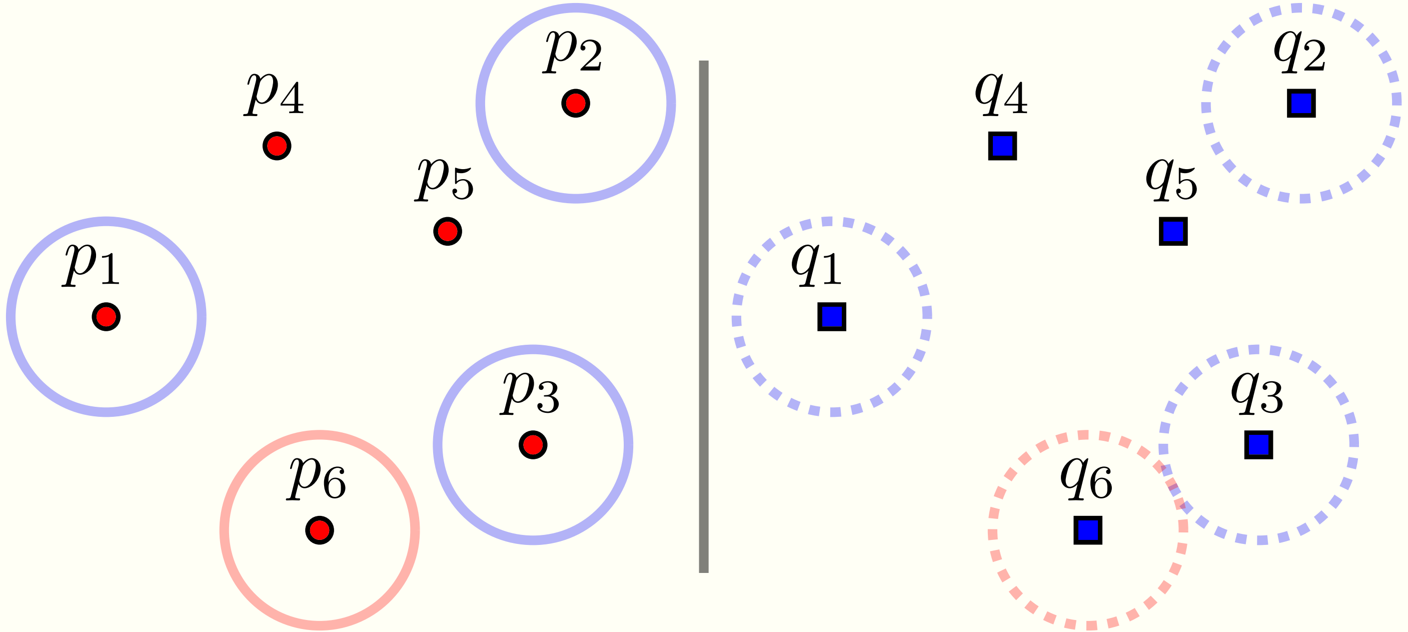

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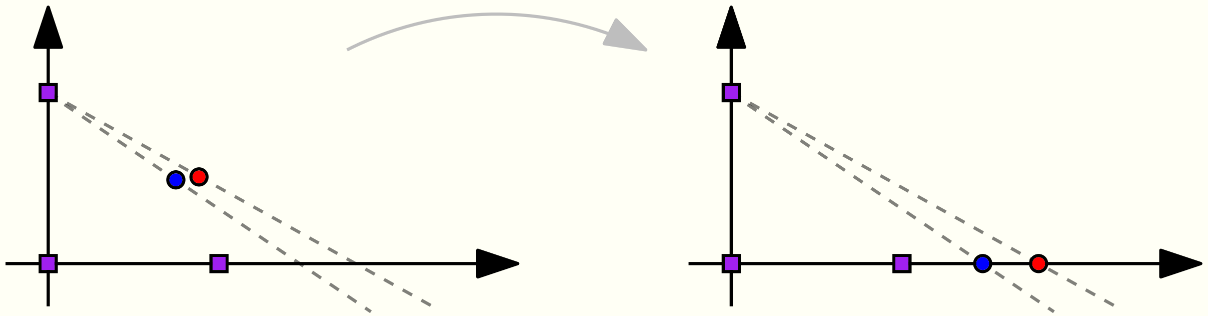

From there, 3 key ingredients :

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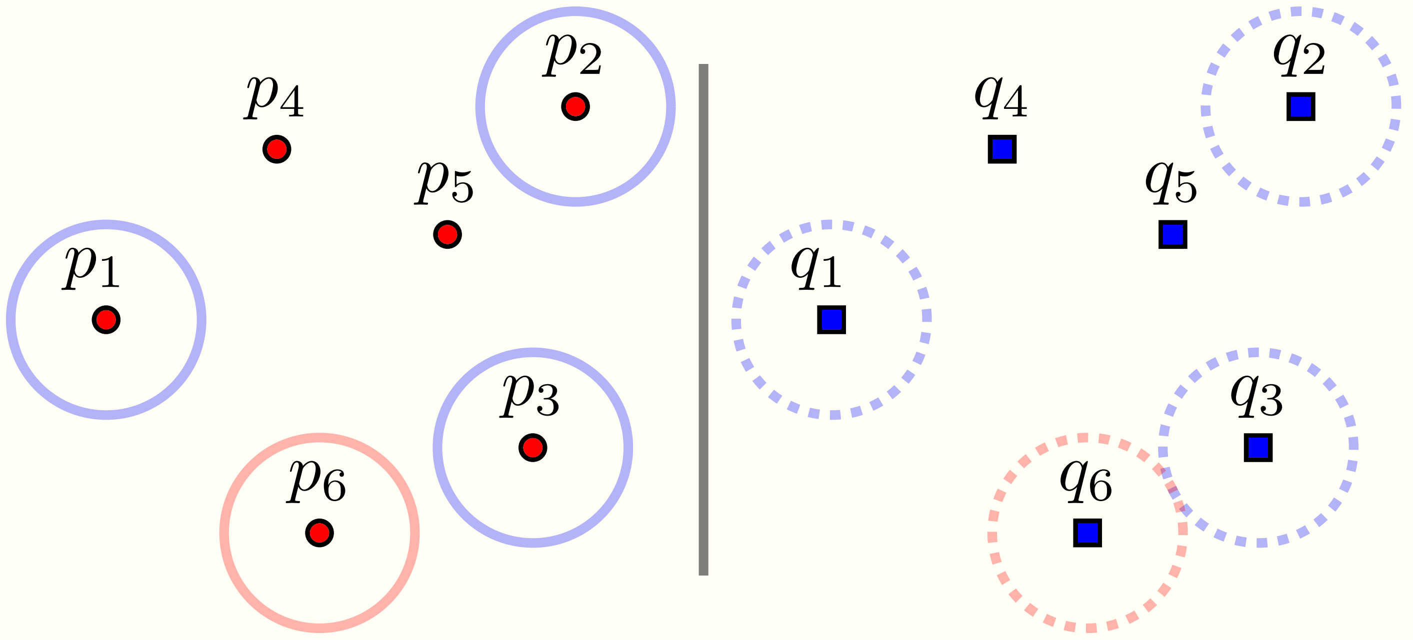

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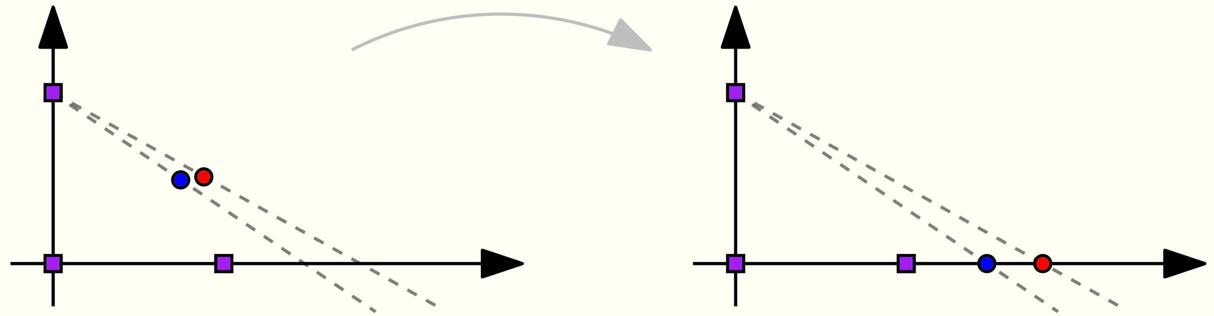

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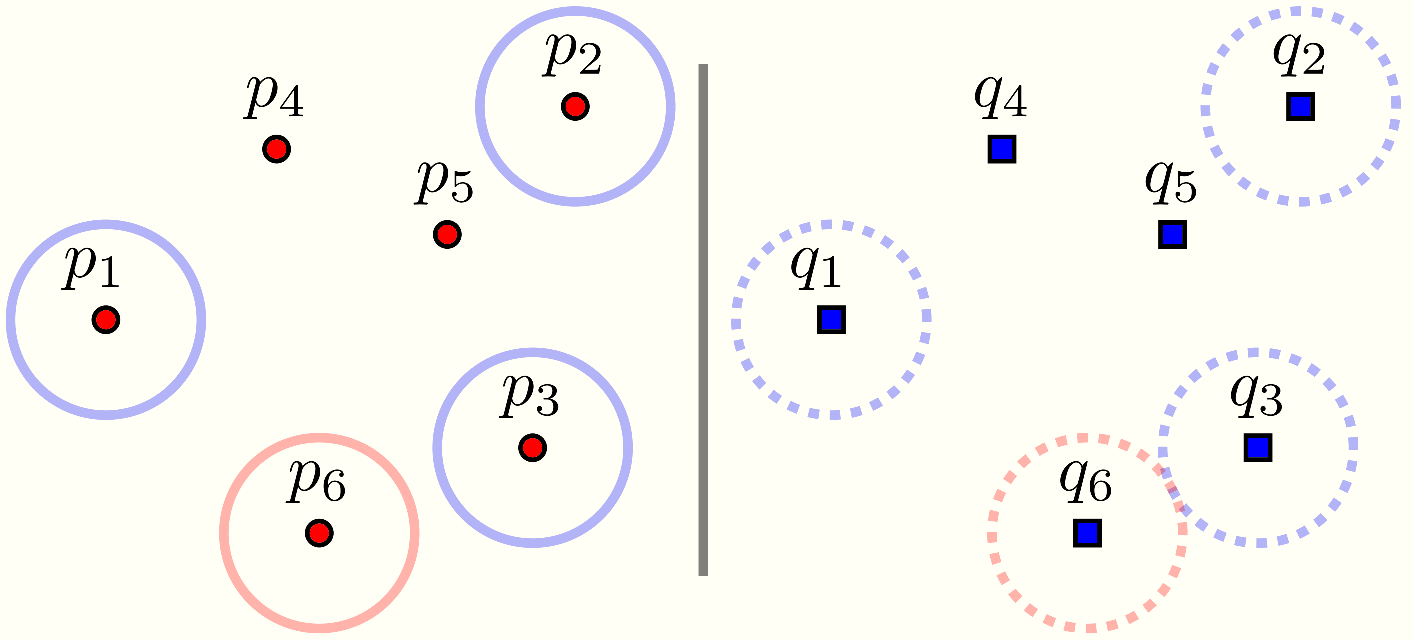

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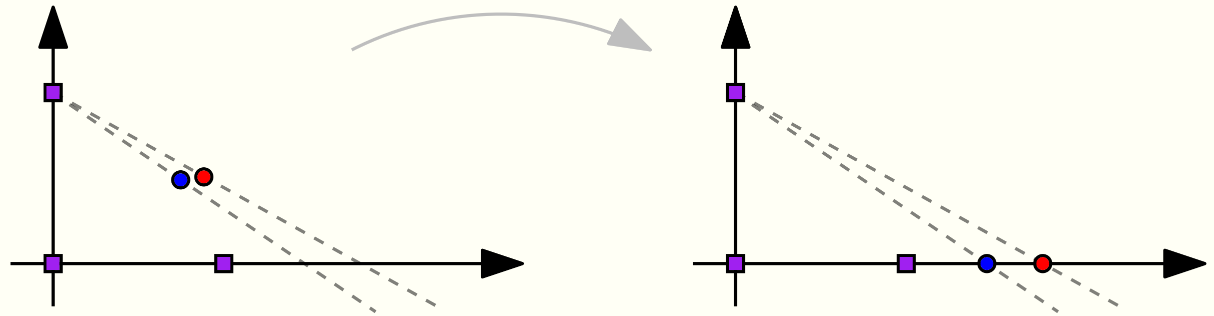

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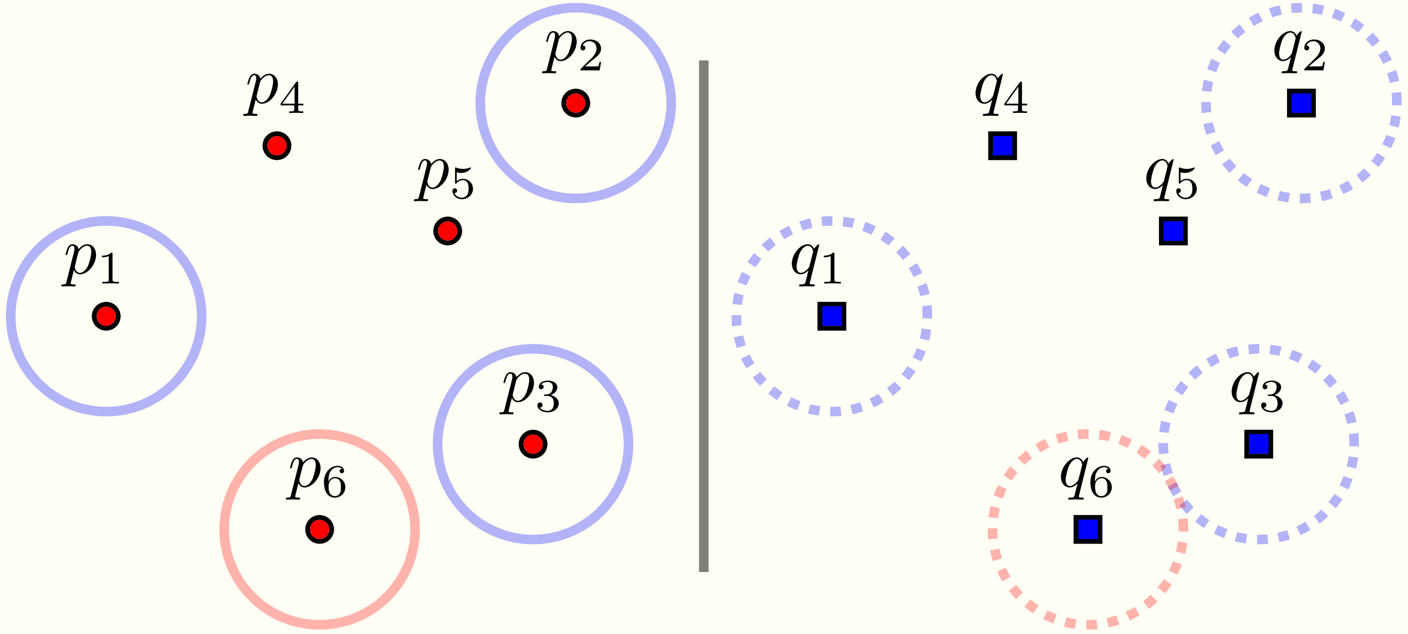

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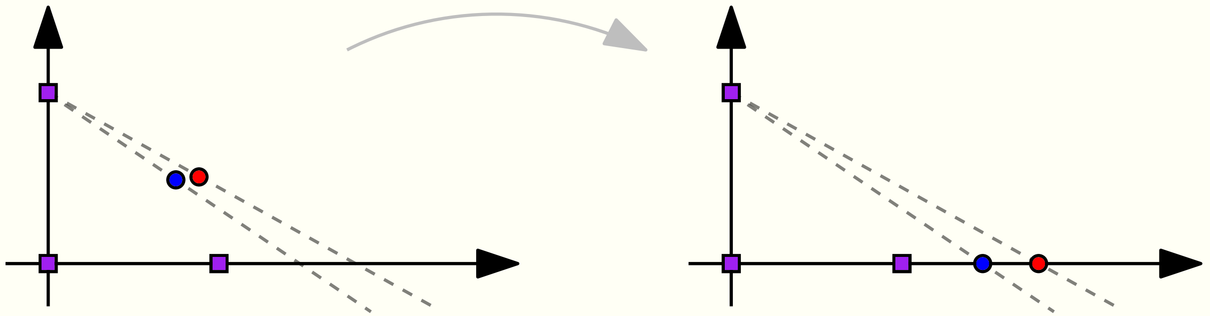

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(1) Parameterizing $\mathbb{R}$ using cross-ratios

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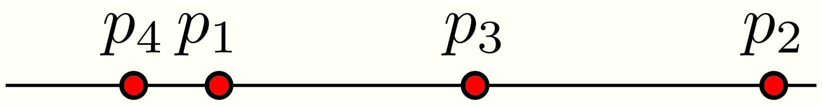

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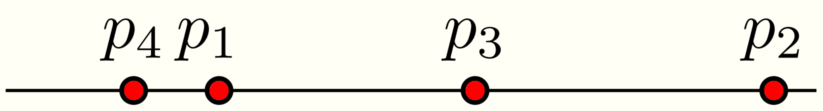

Fix b_0, b_1 and b_∞

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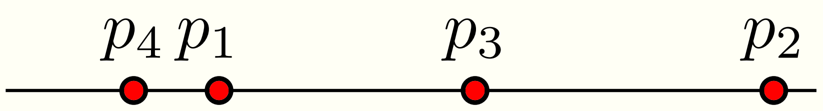

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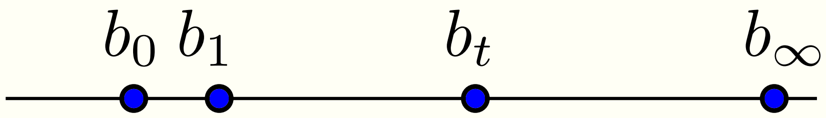

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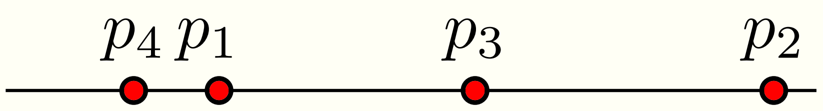

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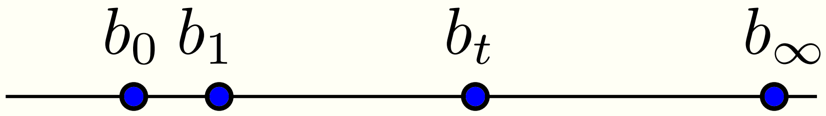

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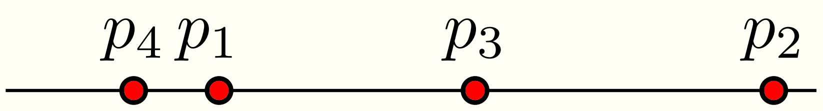

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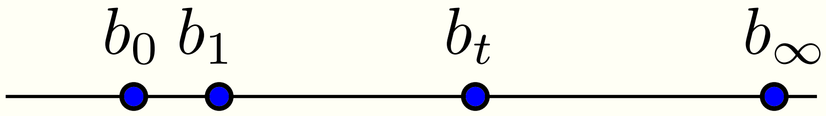

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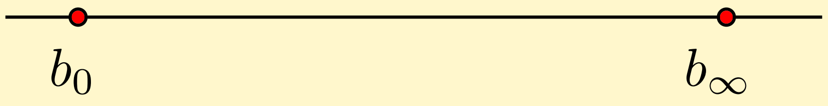

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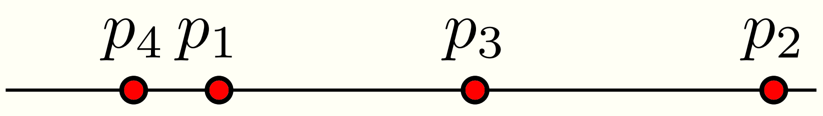

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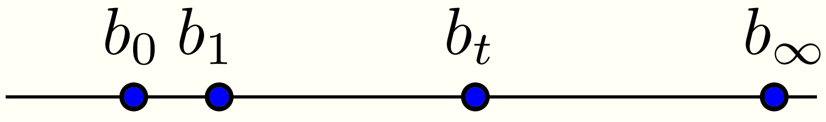

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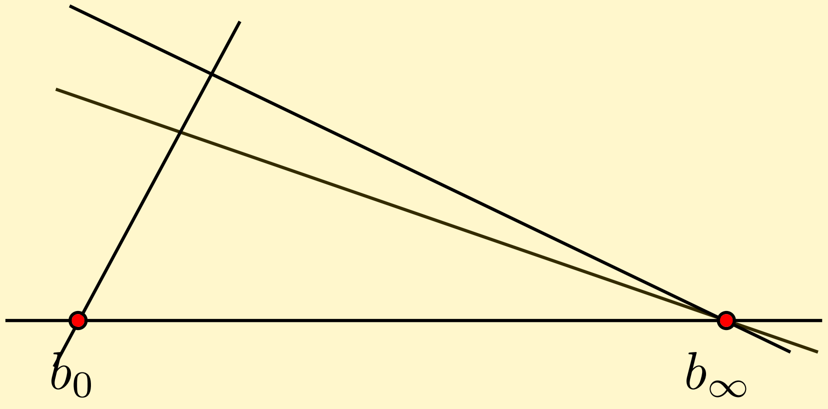

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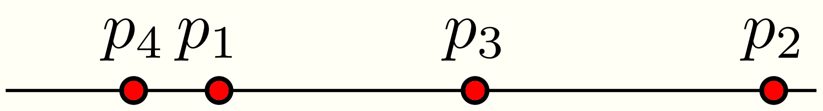

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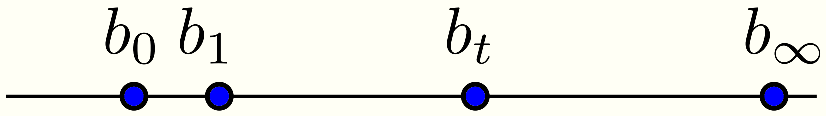

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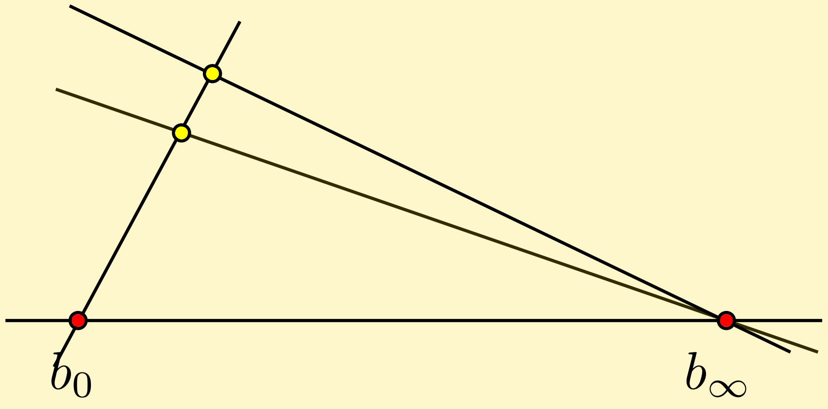

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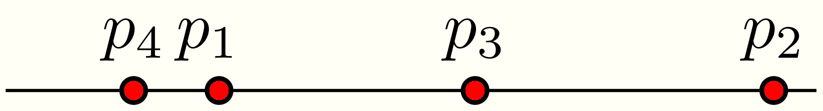

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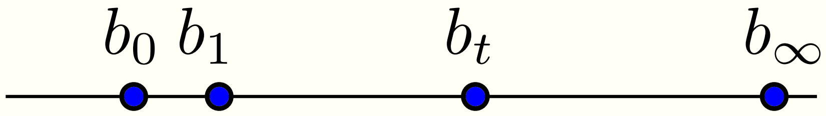

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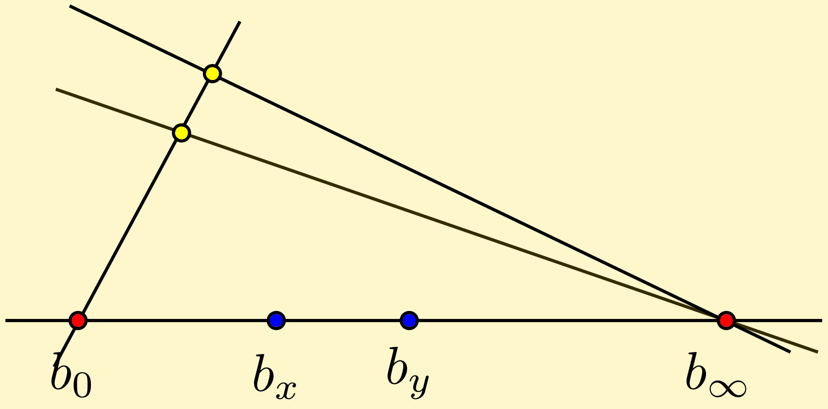

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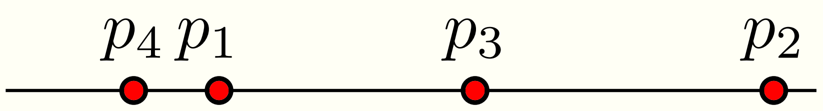

Fix b_0, b_1 and b_∞

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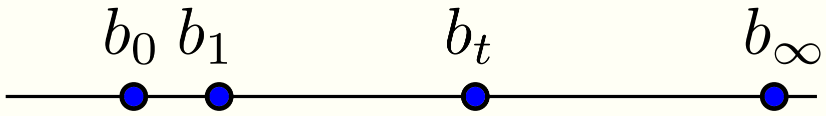
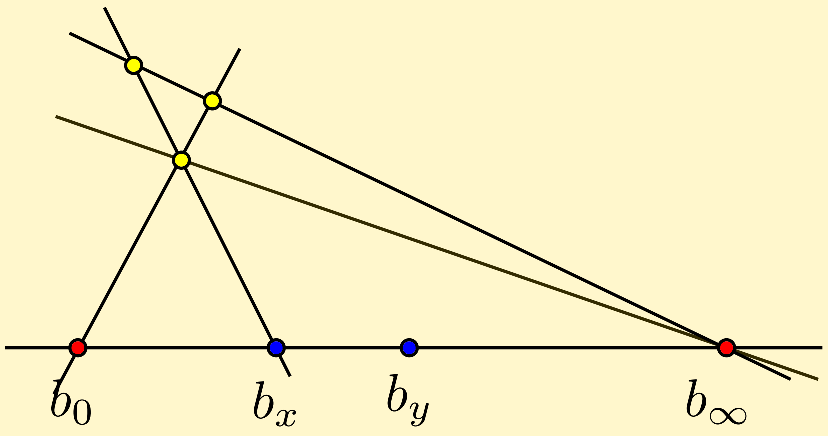


(2) express +,-,*,/ by incidences



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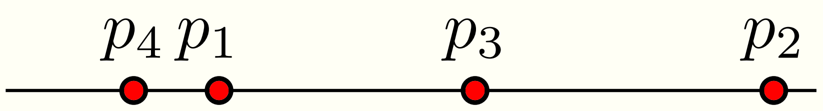

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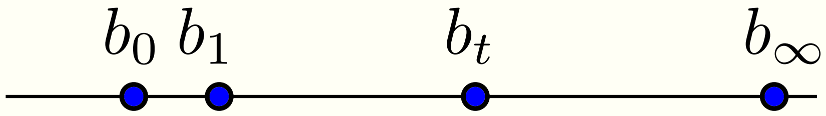

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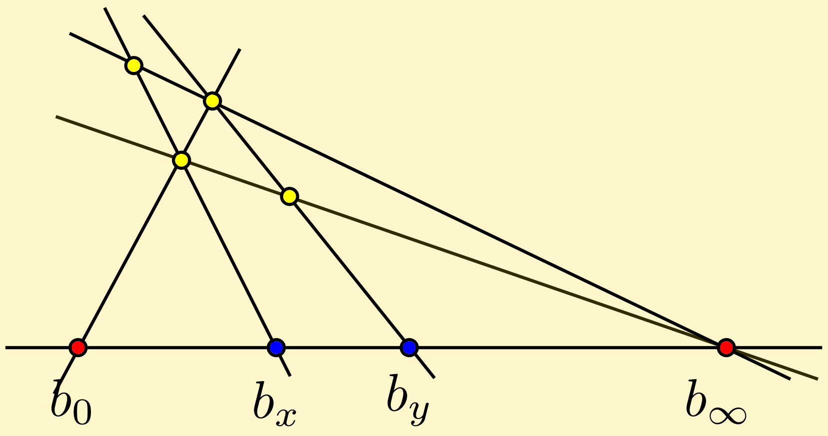

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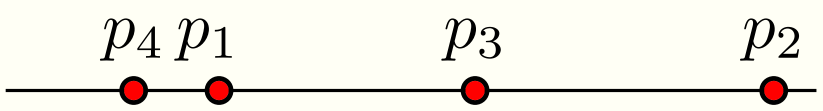

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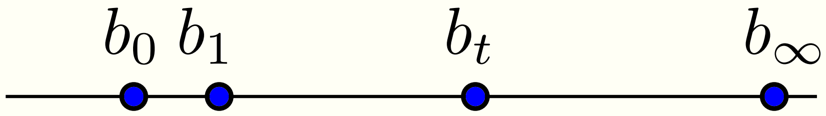

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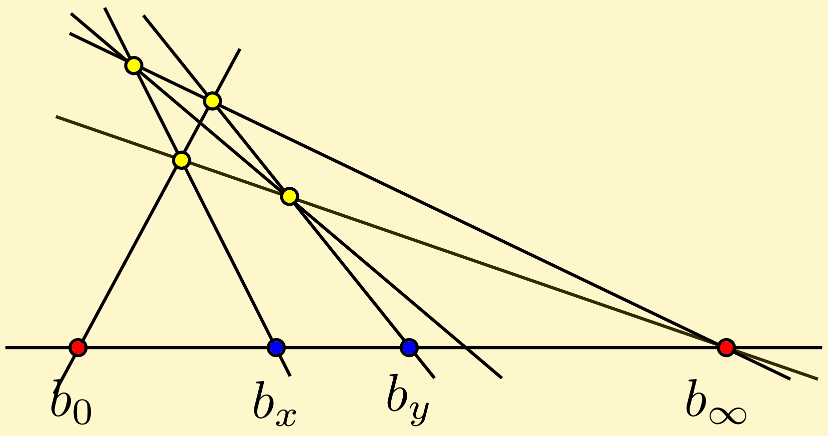

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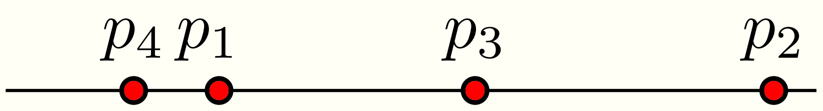

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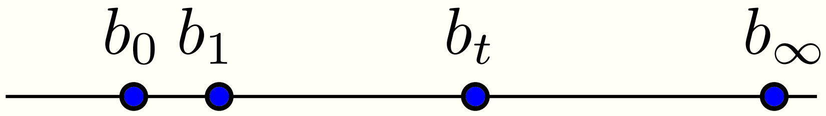

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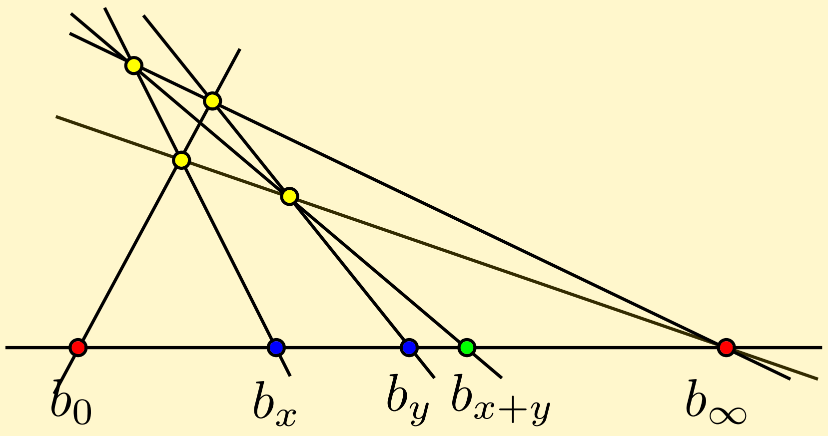

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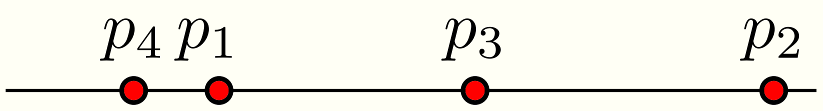

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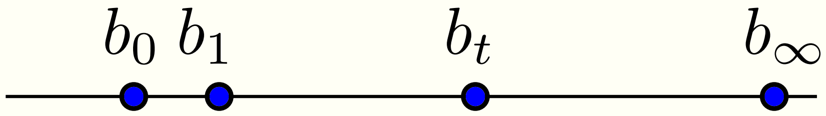

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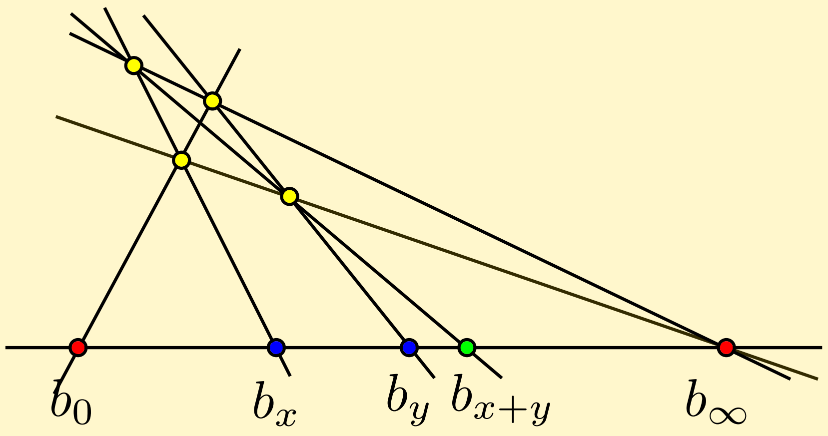

⇝ sequence of **alignment** conditions such that in every **realization** the points on the real line are **projectively equivalent**.

(1) Parameterizing $\mathbb{R}$ using cross-ratios

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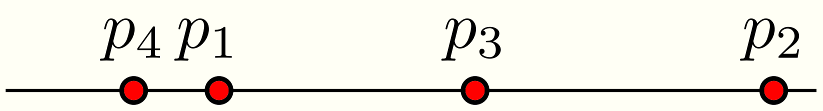

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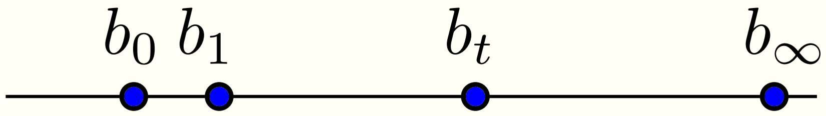

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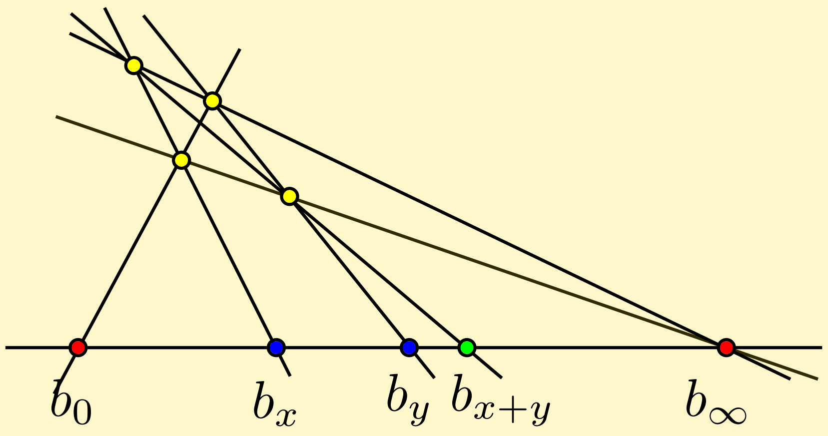

$\rightsquigarrow$ sequence of **alignment** conditions such that in every **realization** the points on the real line are **projectively equivalent**.

(3) Scatter !

(1) Parameterizing $\mathbb{R}$ using cross-ratios

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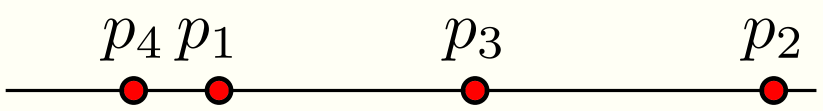

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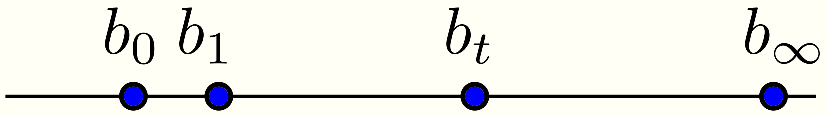

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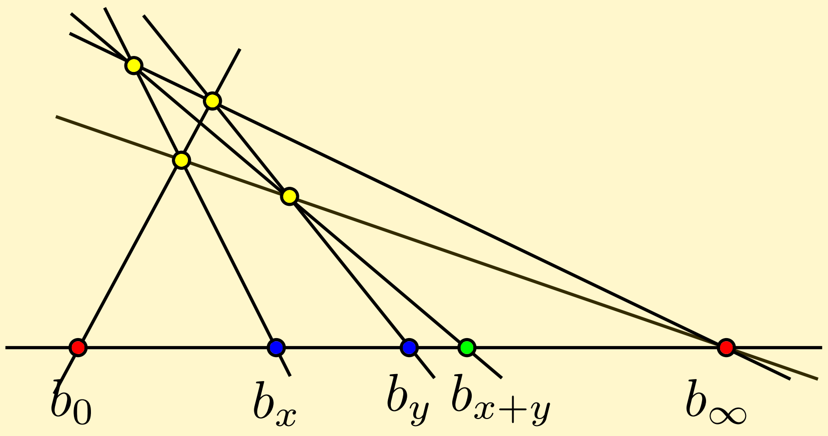

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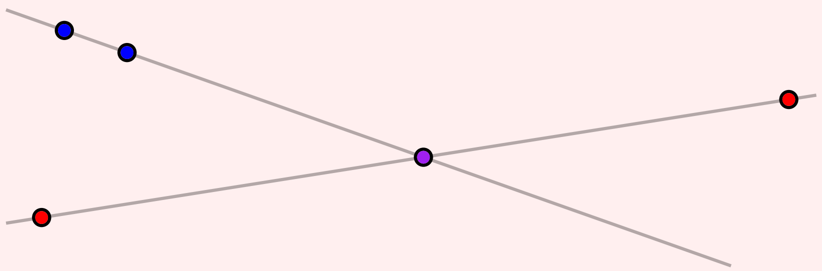

(1) Parameterizing $\mathbb{R}$ using cross-ratios

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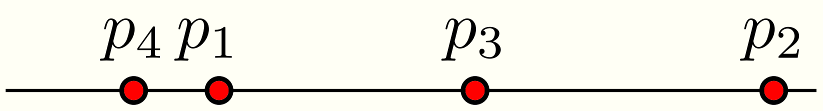

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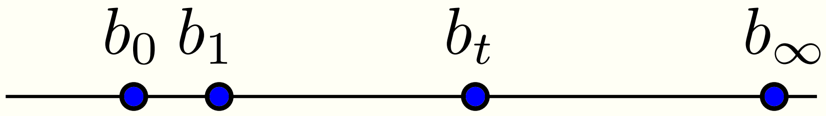

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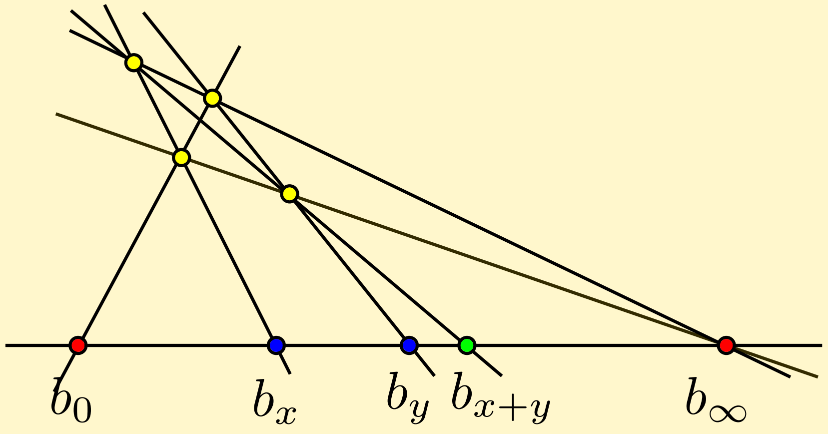

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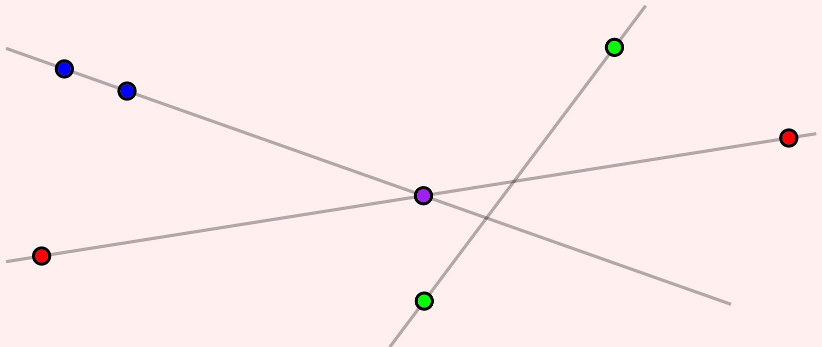

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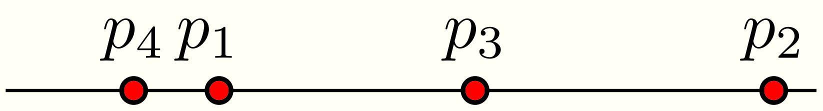

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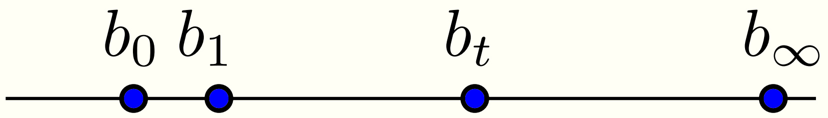

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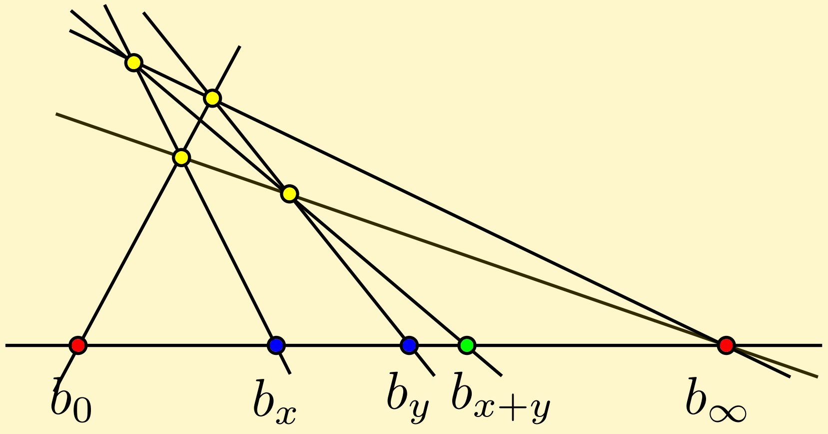

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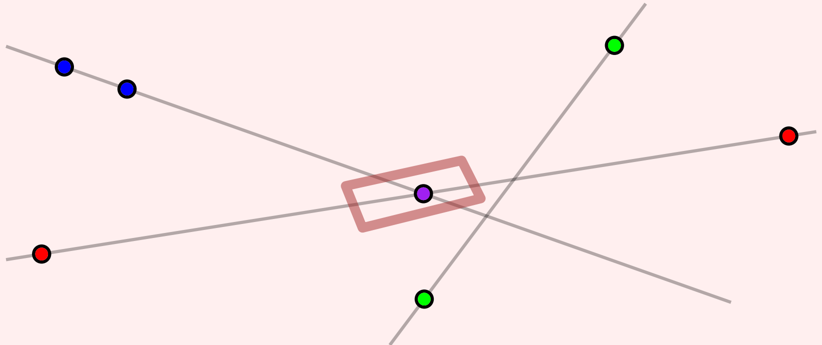

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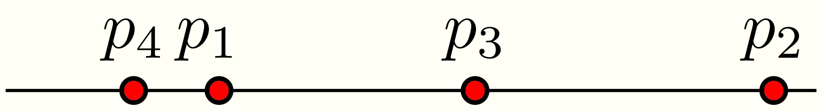

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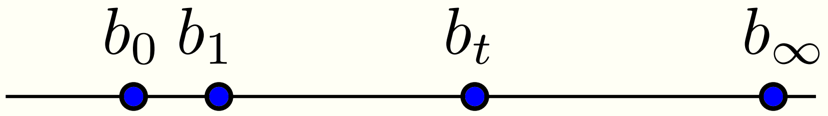

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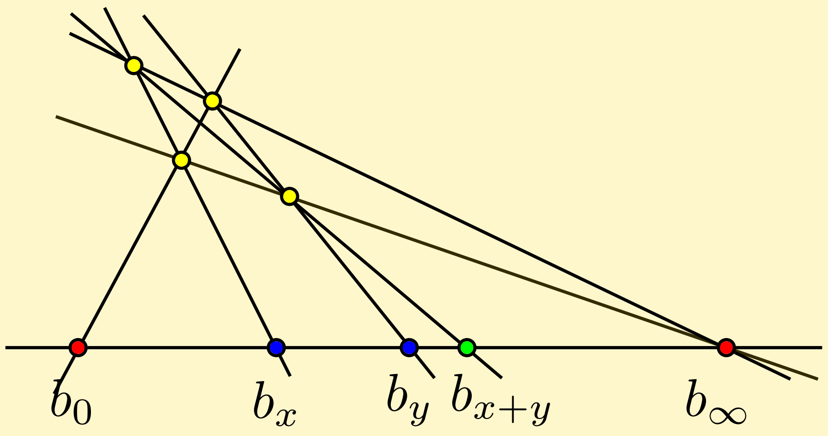

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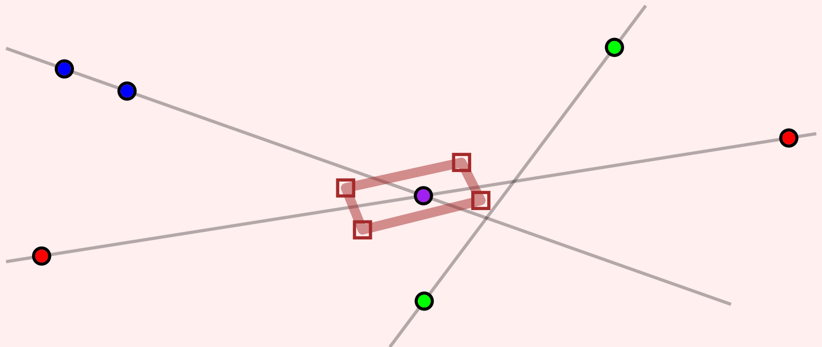

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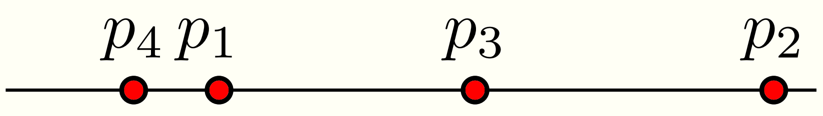

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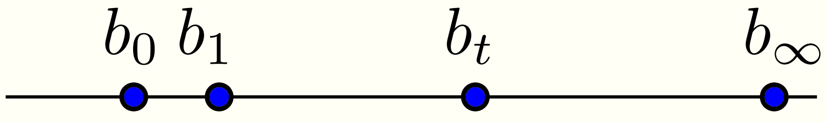

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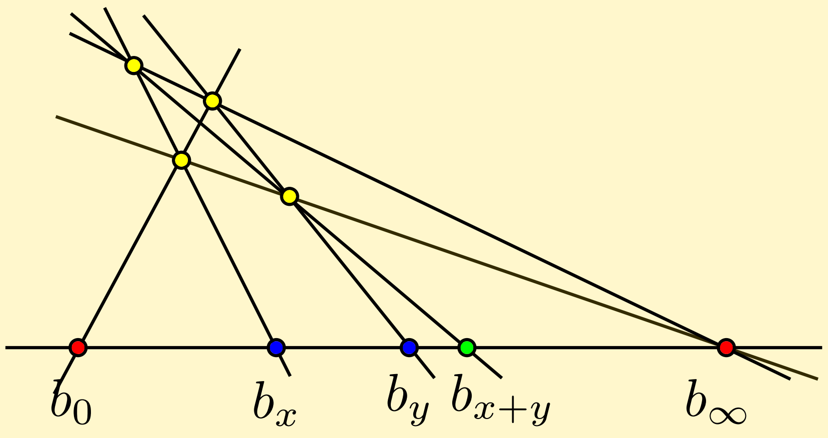

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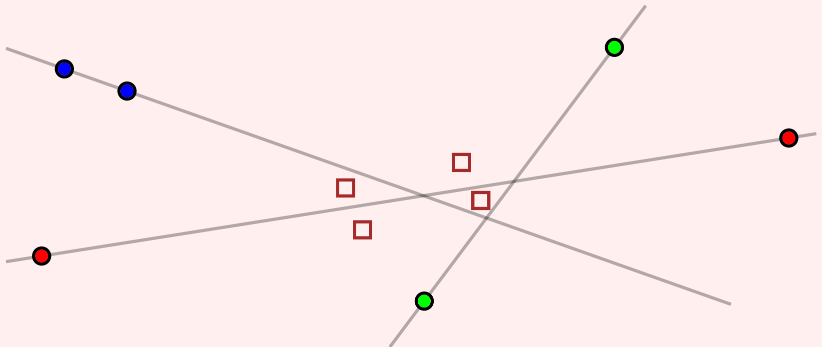
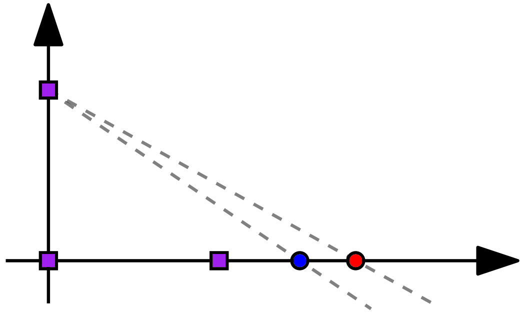


from $\mathbb{R}$ to $\mathbb{RP}^1$



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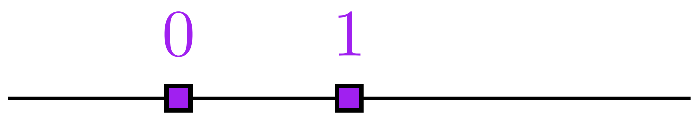

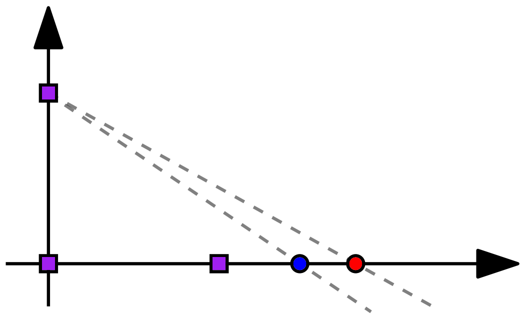

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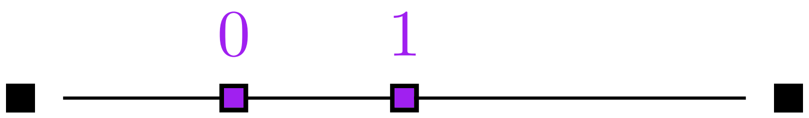

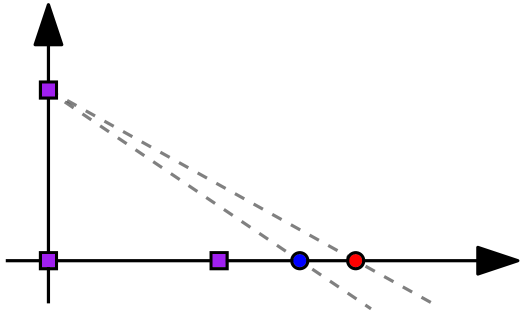

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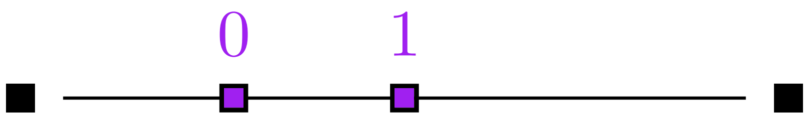
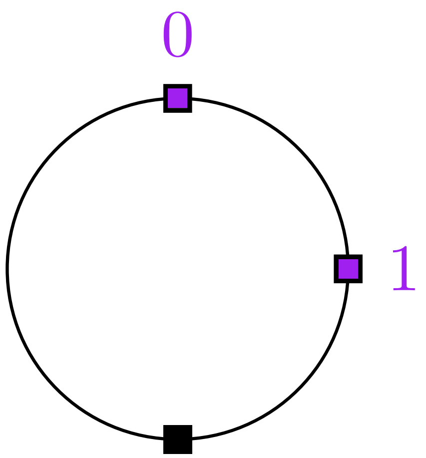
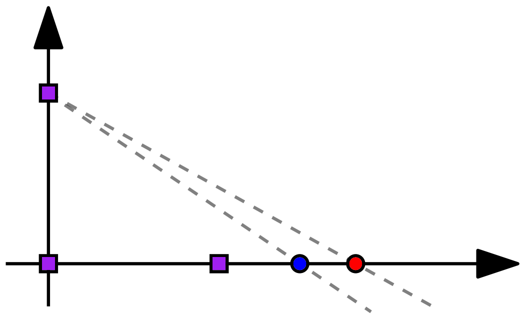
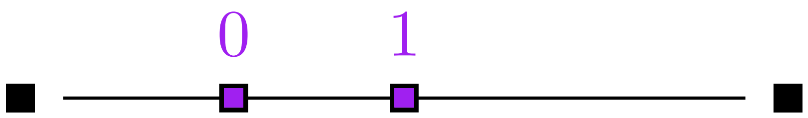
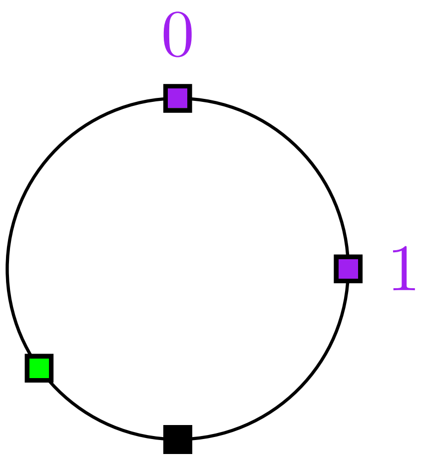


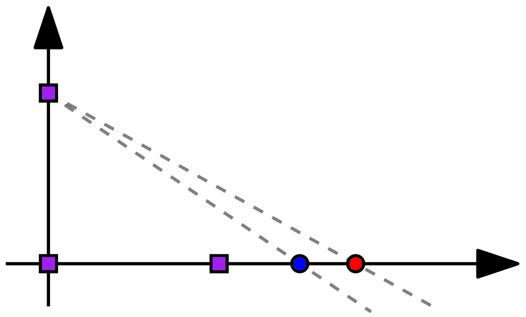




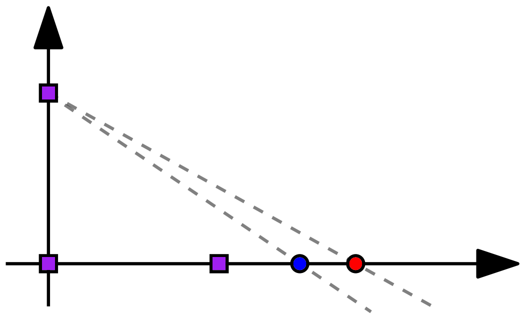
from $\mathbb{R}$ to $\mathbb{RP}^1$

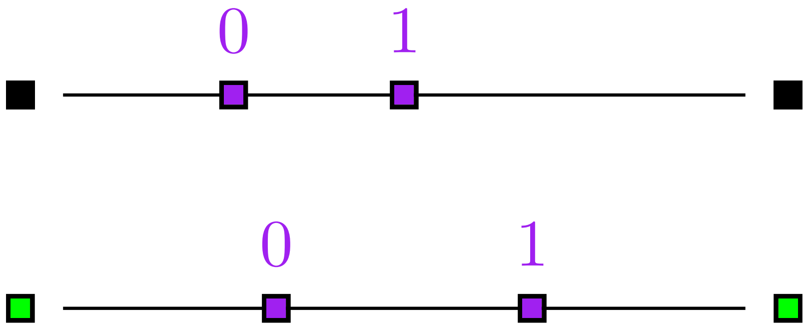


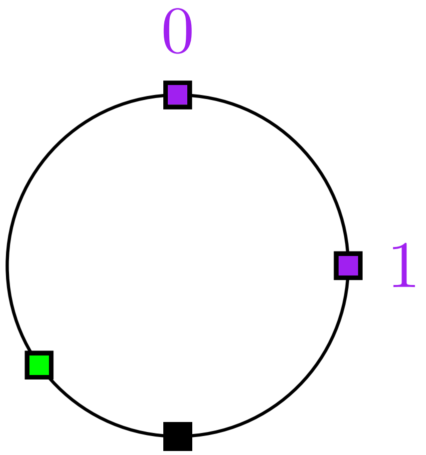




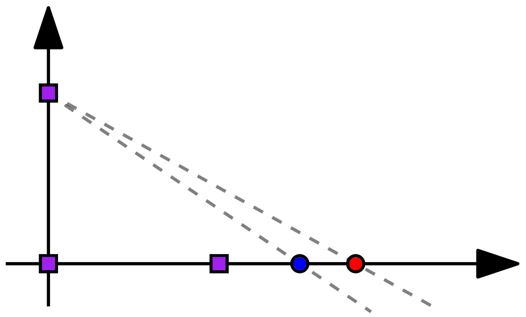
from $\mathbb{R}$ to $\mathbb{RP}^1$

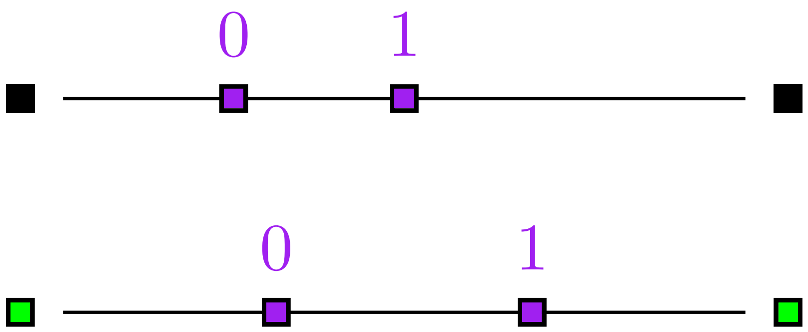


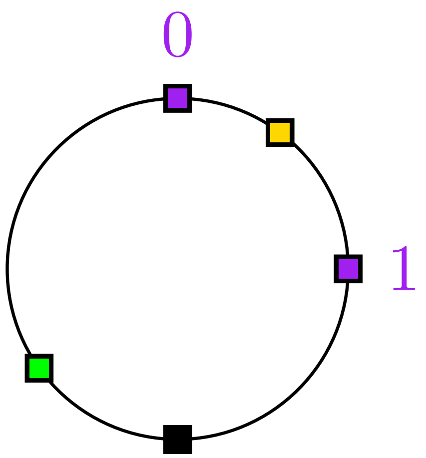




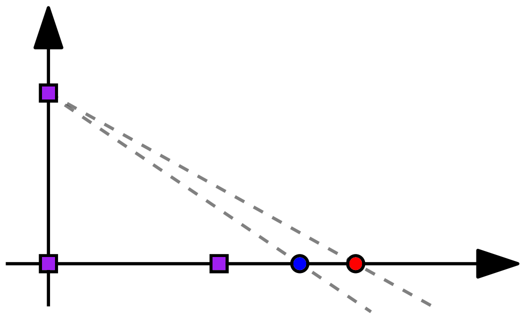
from $\mathbb{R}$ to $\mathbb{RP}^1$

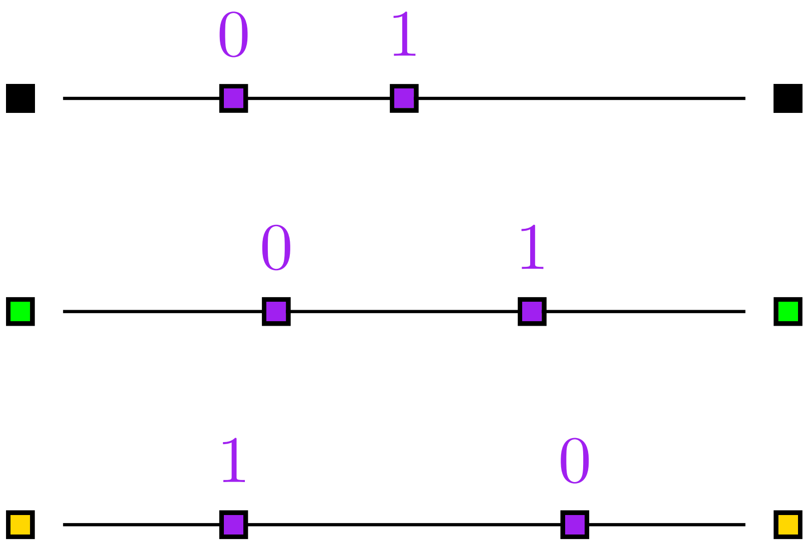


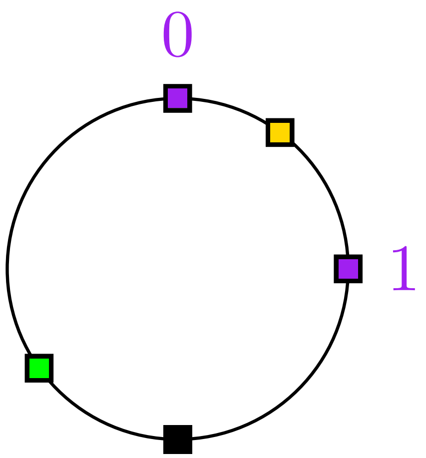




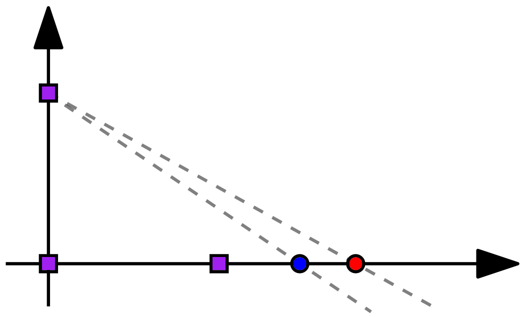
from $\mathbb{R}$ to $\mathbb{RP}^1$

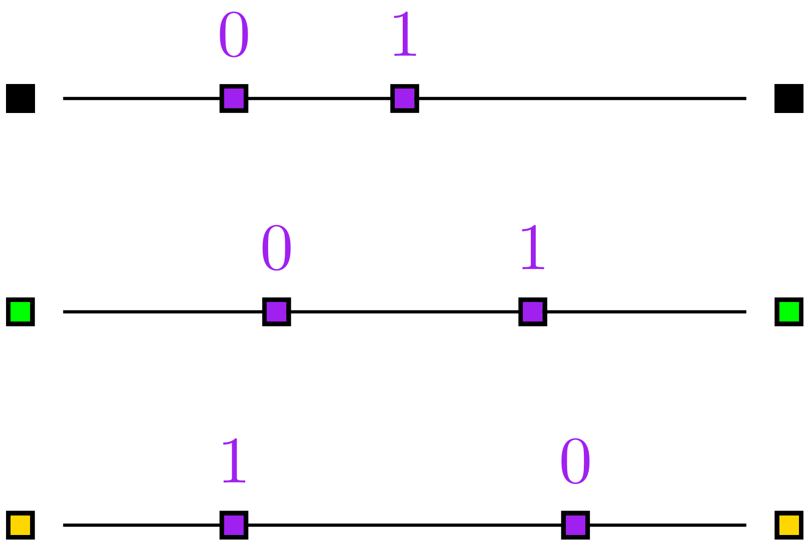


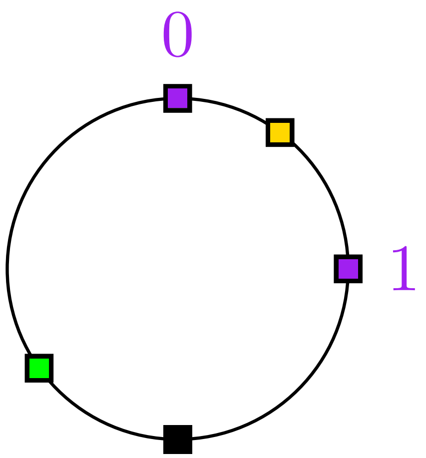


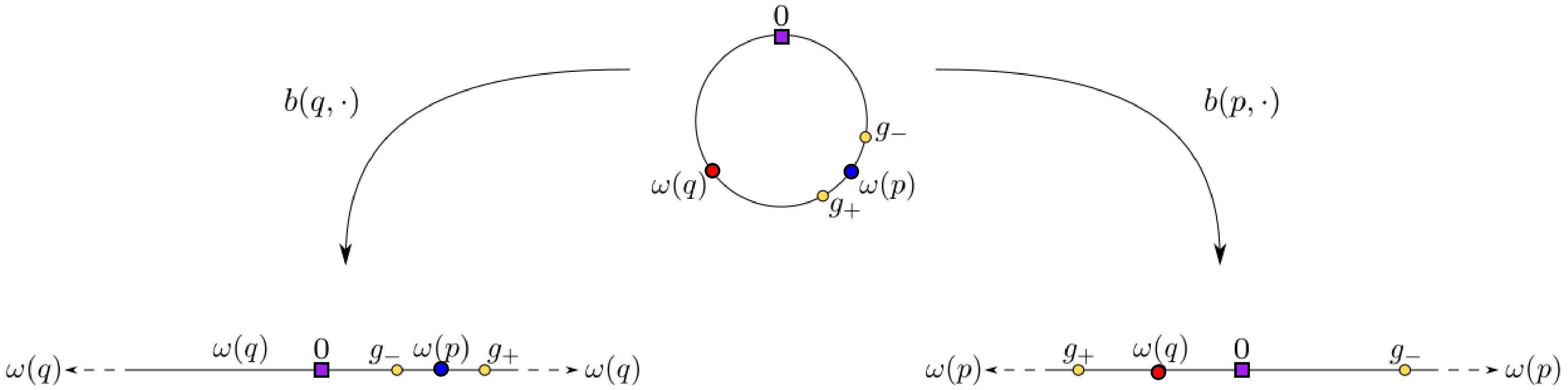


from $\mathbb{R}$ to $\mathbb{R}\mathbb{P}^1$

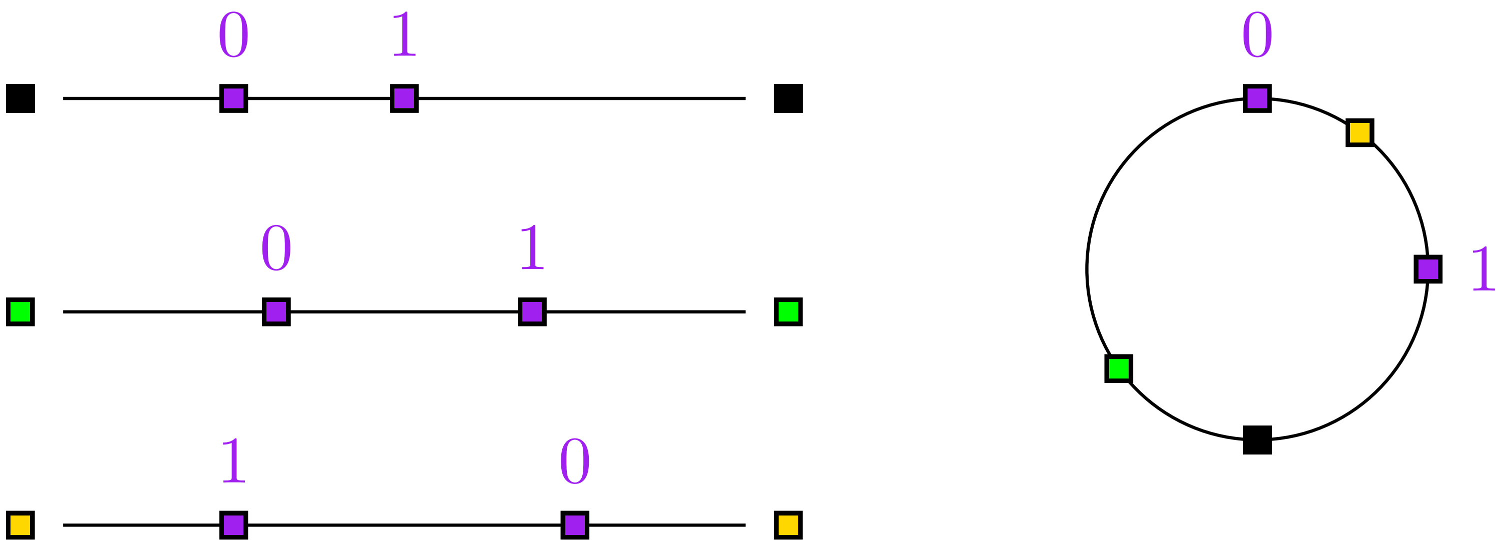


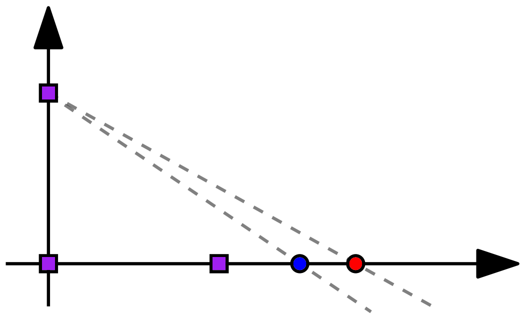


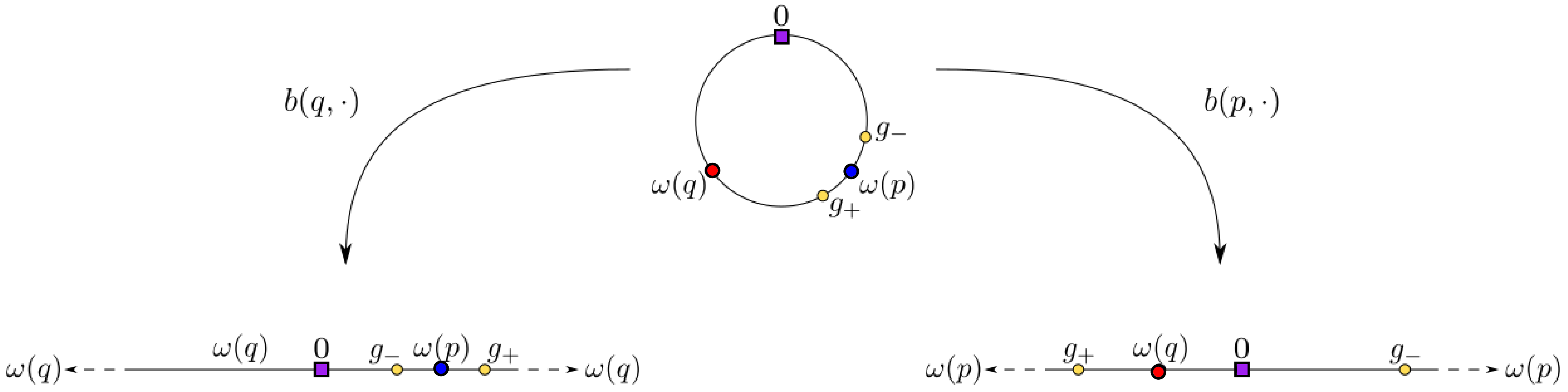




from $\mathbb{R}$ to $\mathbb{RP}^1$







Separate • and • by two points.

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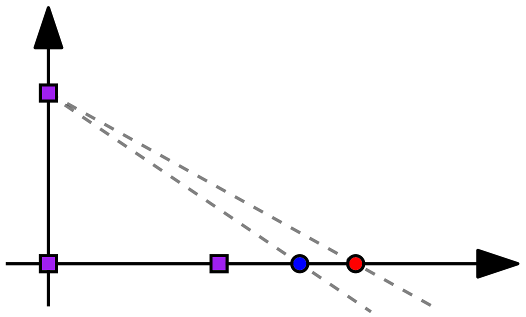
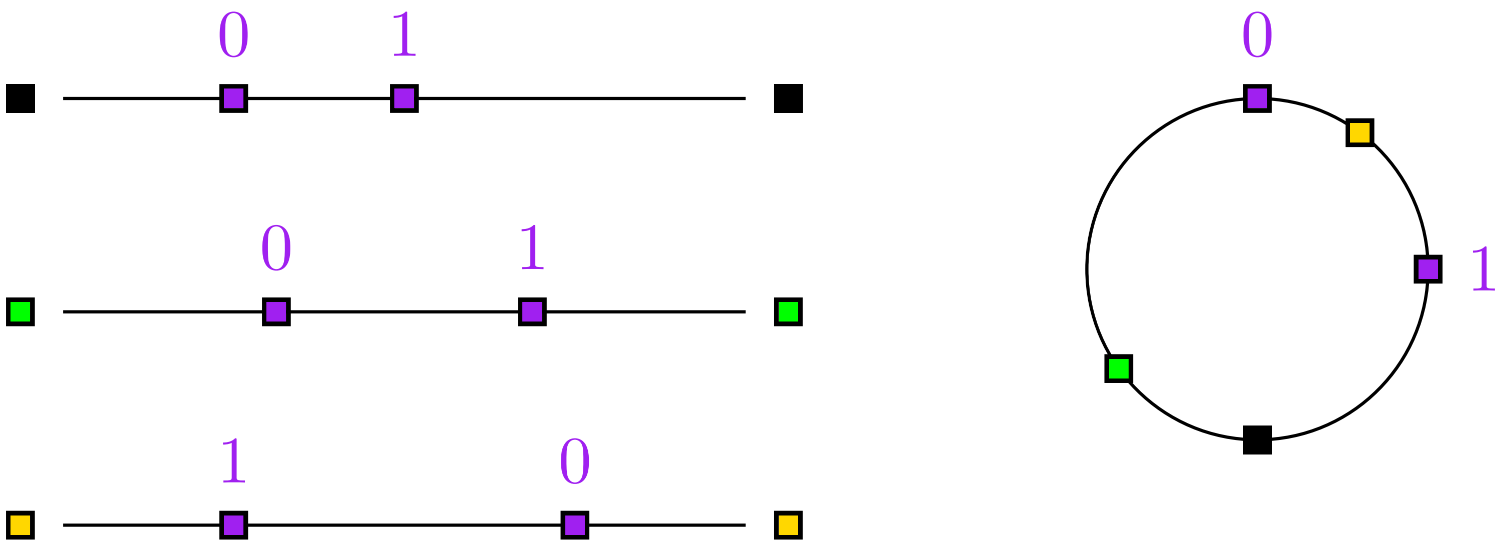
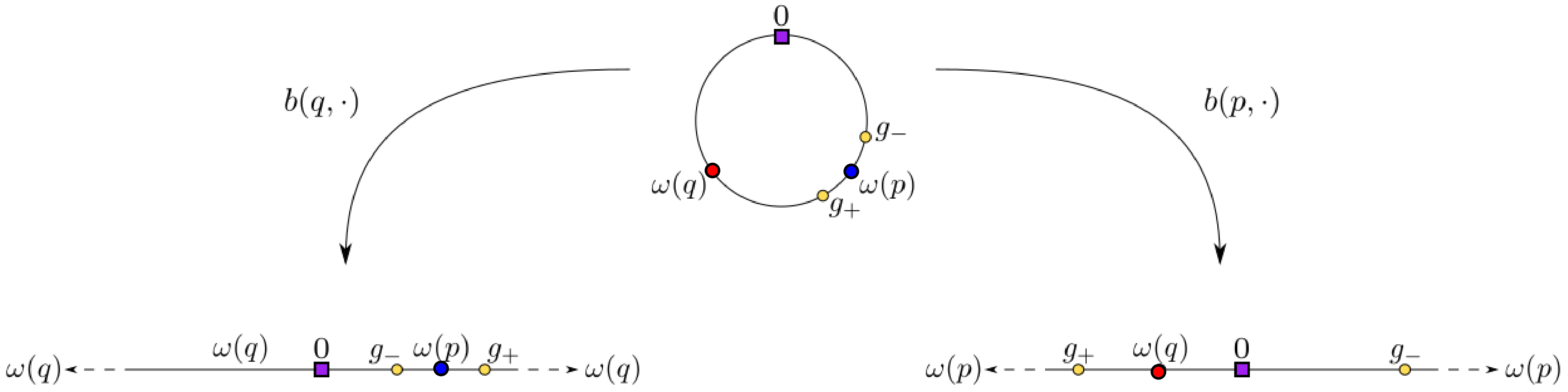
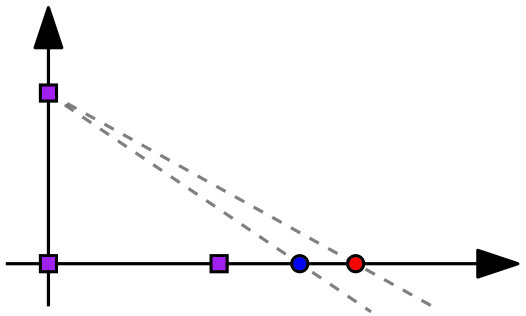


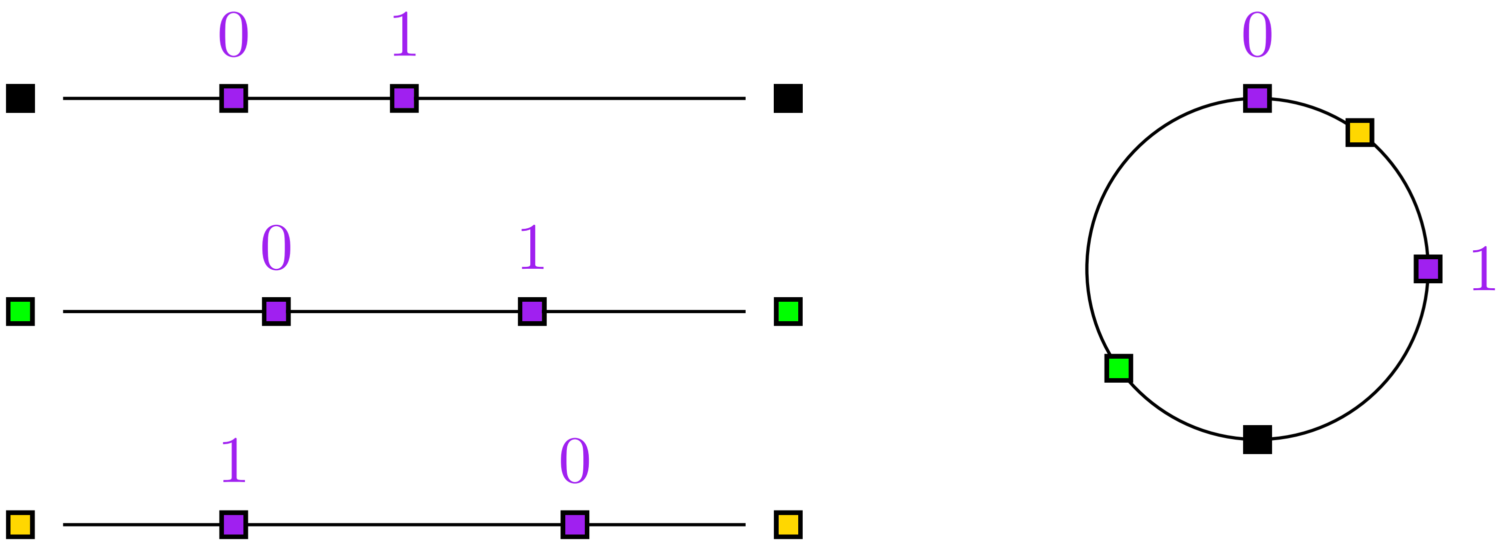


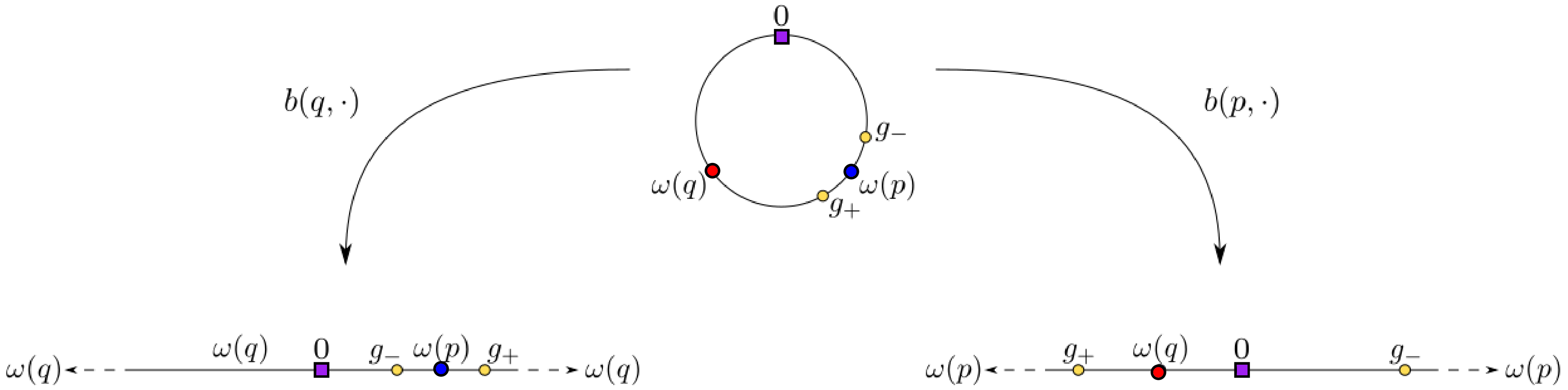


Separate • and • by two points. Construct these points (Von Staudt).

from $\mathbb{R}$ to $\mathbb{RP}^1$







Separate • and • by two points. Construct these points (Von Staudt). Analyze wrt ε...

Many open questions...

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Thank you for your attention !