Lifting maps between graphs to embeddings

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Topological liftings

Let $f\colon X \to Y$ be a (smooth / piecewise linear / continuous) map between topological spaces. A *topological lifting* (to an embedding) is an embedding

$$F\colon X \hookrightarrow Y \times \mathbb{R}$$

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This concept is related to a classical question in topology: can a space X be embedded into some $\mathbb{R}^n$?

It is natural to put additional restrictions on embeddings $\implies$ the *liftability problem* for maps $X \to \mathbb{R}^{n-1}$.

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Theorem (Poénaru, 1979; Carter–Saito, 1998; G., 2025)

The liftability problems for smooth immersions and for non-degenerate piecewise-linear maps between polyhedra reduce to the case of non-degenerate piecewise-linear maps between graphs.

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Geometric intuition

Given a graph morphism $f\colon G \to H$, we want to obtain an embedding by:

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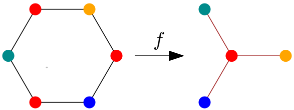
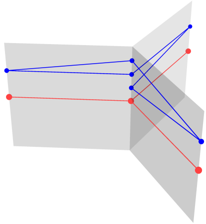


A map.

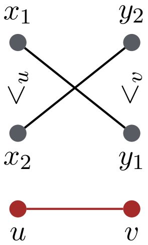


A lifting.

Combinatorial liftings

Suppose $f\colon G \to H$ is a morphism of graphs. A *combinatorial lifting* is a collection of linear orders $\{ <_p | \, p \in V(H) \}$ on the fibers $f^{-1}(p)$ without crossings.

A *crossing* occurs when there exist edges $x_1y_1, x_2y_2 \in E(G)$ such that $f(x_1)f(y_1) = f(x_2)f(y_2) = uv \in E(H)$, but $x_1 >_u x_2$ and $y_1 <_v y_2$.



Combinatorial liftings

Theorem (G., 2025)

For any graph morphism $f : G \to H$, there is a bijection between combinatorial liftings and isotopy classes of topological liftings (of $|f|$).

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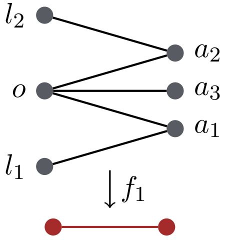
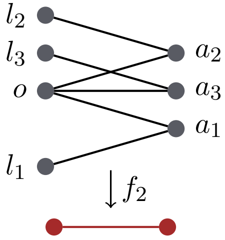


Liftable map



Non-liftable map

Problems

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When we consider maps to path graphs, these problems are known as the *level planarity* and *constrained level planarity* problems, respectively.

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- *Constrained level planarity*: NP-complete even when G is a union of path graphs (Klemz, Rote, 2019) and for 4 levels (Blažej et al, 2024).

Graphs of pairs and 2-obstructors

Let $f\colon G \to H$ be a graph morphism. The *graph of pairs* $G_f^{(2)}$ consists of:

- vertices: pairs (a, b) of distinct vertices of G with $f(a) = f(b)$;
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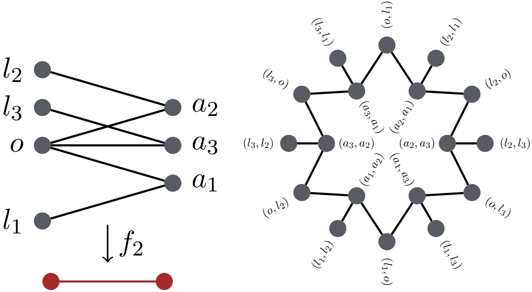
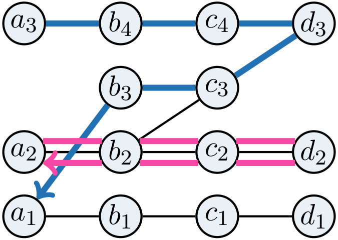
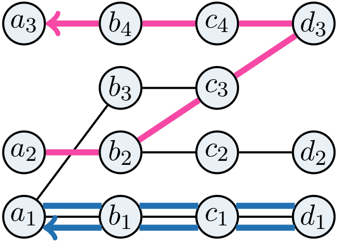


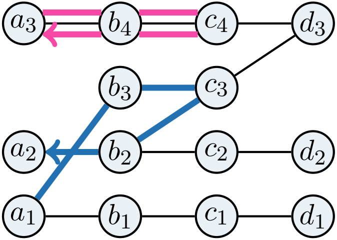
2-obstructors: Siekłucki's example



First part



Second part



Third part

2-obstructor condition

Conjectures

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- It is not difficult to deduce this from the existing Hanani–Tutte-style results on level planarity. Also, there are two alternative proofs: for generic maps $T \to P$ (G., 2025), and for arbitrary maps $T \to P$ (not yet published).
- There exists a non-liftable map $G \to T$ satisfying the 2-obstructor condition (G., 2025).

Geometry of $G_f^{(2)}$ and "moves" on liftings

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1. Liftings of these maps can be changed by certain "moves".
2. The geometry of $G_f^{(2)}$ tells us when such moves are possible.
3. This leads to a family of conditions extending the 2-obstructor condition.

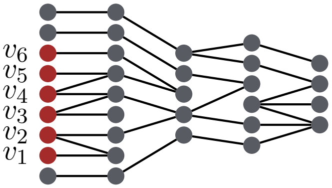
Geometry of $G_f^{(2)}$ and “moves” on liftings: example

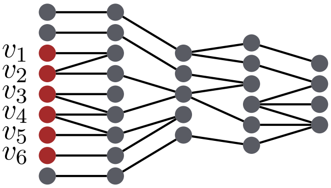
Theorem

Let $f\colon T \to P$ be a map from a tree to a path graph, and let F be its lifting.

- Consider a segment $[a, b]$ in the first fiber $f^{-1}(0)$.
- Let l, h be the minimal and maximal vertices in the last fiber $f^{-1}(\max P)$.

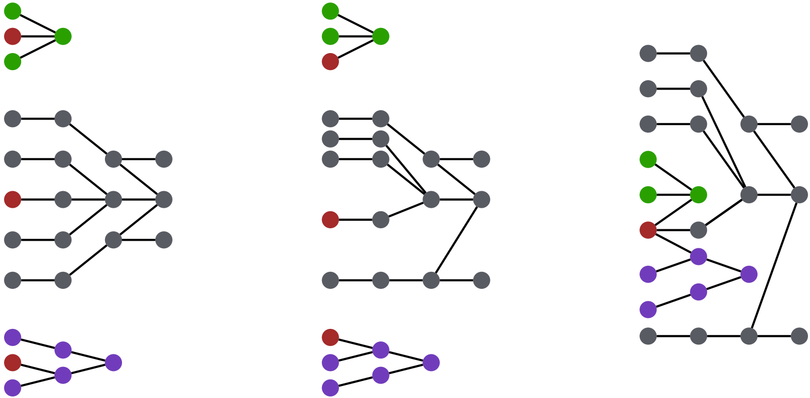
Then $[a, b]$ can be flipped without changing the order in the last fiber if (a, b) and (l, h) lie in different components of $T_f^{(2)}$.





Geometry of $G_f^{(2)}$ and "moves" on liftings: combining liftings

Liftings can be *combined* using these "moves" by identifying selected leaves in the first fibers. The result is a valid lifting *iff* the 2-obstructor condition holds for the combined map.



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1. We provide a constructive, self-contained proof that the 2-obstructor condition is complete for trees to path graphs. This also yields a polynomial algorithm for constructing a lifting: not the most efficient, but relatively simple to implement.

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2. Using these "moves", the constrained liftability problem for trees to path graphs (with total orders in first and last fibers as constraints) can be solved in polynomial time.
3. We hope this approach can be generalized to other settings, e.g., maps from arbitrary graphs to path graphs, or maps to trees.

References

1. NP-completeness in general, relations to topology and approximability by embeddings:

   A. G. "Lifting maps between graphs to embeddings", Essays on Topology: Dedicated to Valentin Poénaru, 2025. *arXiv:2404.12287*.

2. Overview of results on level planarity mentioned in my talk:

   Simon D. Fink, Matthias Pfretzschner, Ignaz Rutter, Peter Stumpf, "Level Planarity Is More Difficult Than We Thought", *arXiv:2409.01727*.

3. NP-completeness of constrained level planarity:

   Boris Klemz, Günter Rote, "Ordered level planarity and its relationship to geodesic planarity, bi-monotonicity, and variations of level planarity." ACM Transactions on Algorithms, 15(4):1–25, 2019. *arXiv:1708.07428*.

4. Lifting "moves": WIP.