
Estimating the Persistent Homology of $\mathbb{R}^d$ -valued functions

joint work with Ethan André, Jingyi Li, and Steve Oudot

David Loiseaux

JGA, 16 Octobre 2025



Context

Some datasets convey interesting *geometric* and *topological* information.

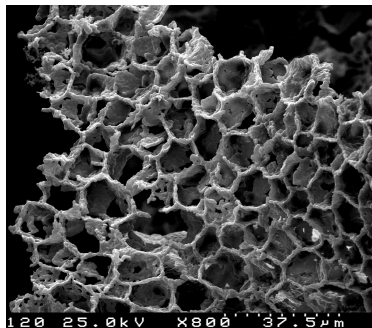


Figure: Porous magnetic hexaferrite foams, Zeiss Microscopy.

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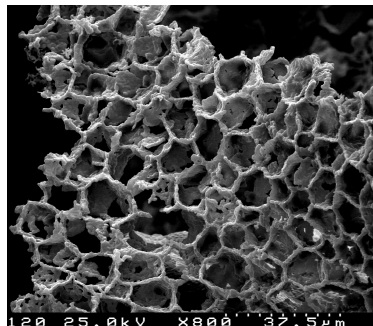


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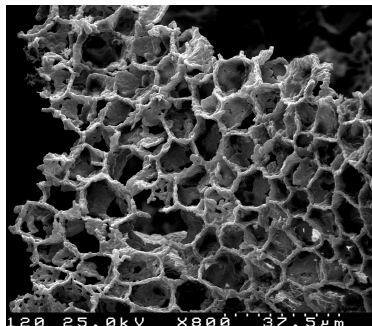


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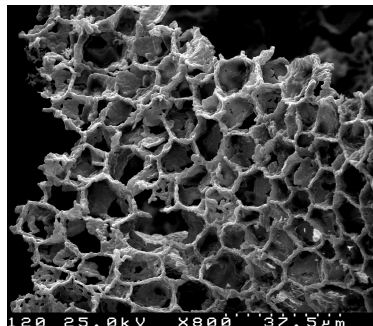


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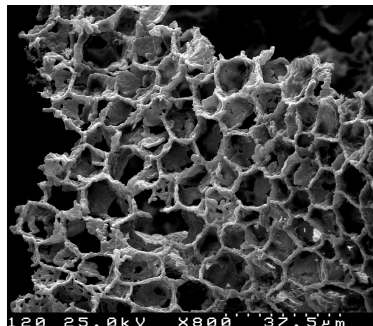


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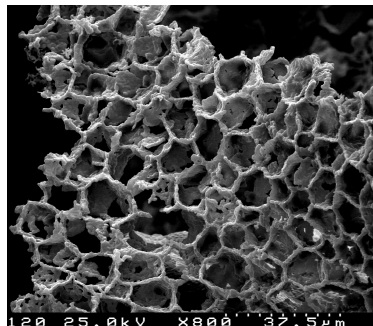


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5. robustness guarantees, e.g., *stability*, *interpretability*.

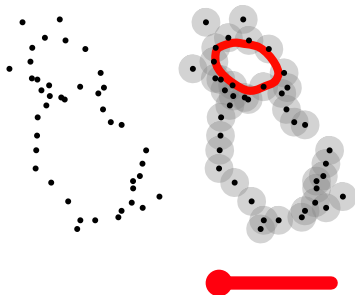
Topological persistence

Čech's filtration



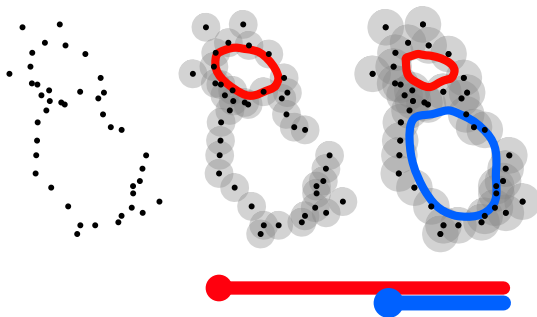
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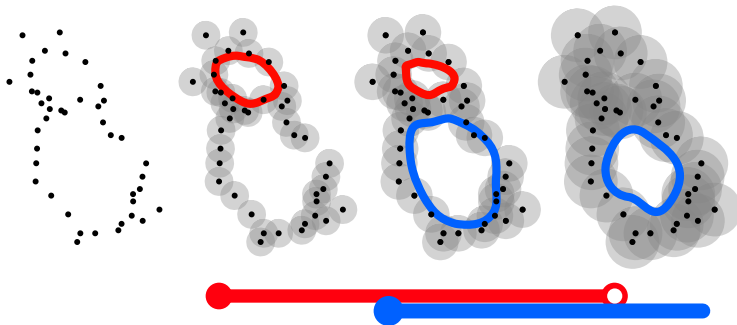
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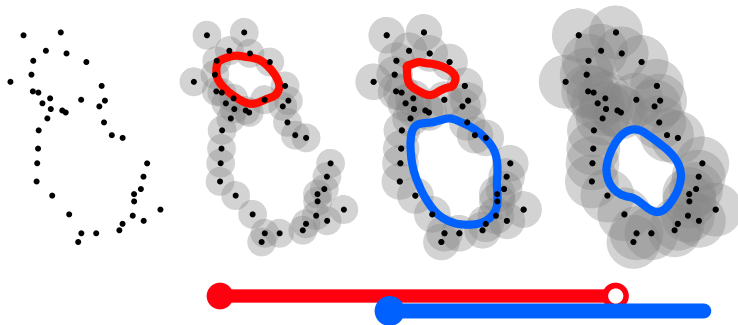
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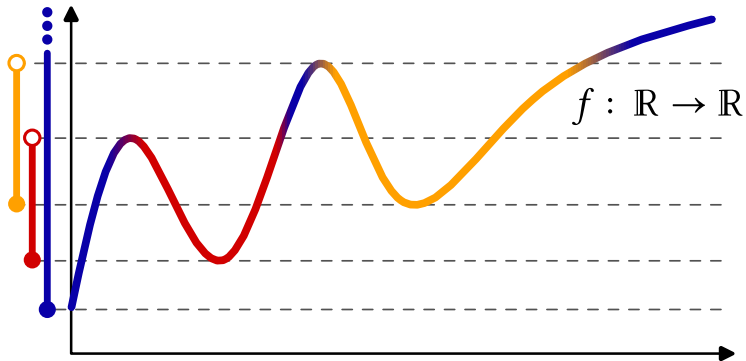
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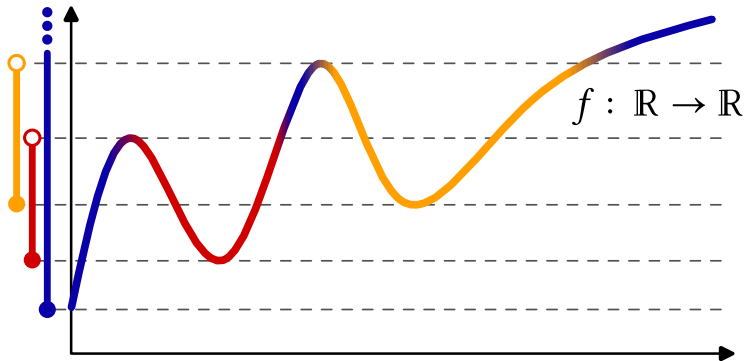


The *persistence barcode* is the multiset of bars, encoding the *lifetimes* of topological structures over time (radius).

Sublevelset example

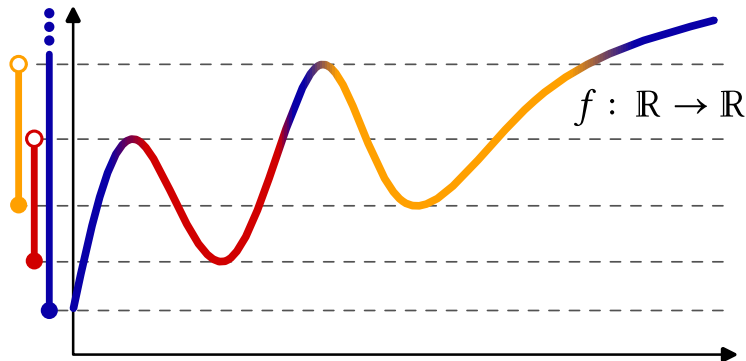


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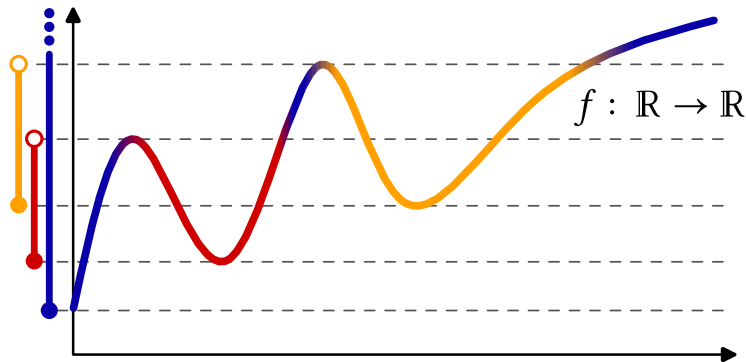
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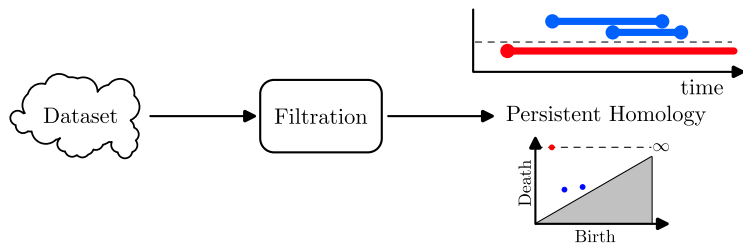
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1. The function $f : X \rightarrow \mathbb{R}$ induces a filtration $F = (f^{-1}(-\infty, t])_{t \in \mathbb{R}}$.
2. The time parameter $t \in \mathbb{R}$ encodes the “geometry” along f .
3. $H_0(F) = k \text{ — blue — } \oplus k \text{ — red — } \oplus k \text{ — orange —}$ encodes the “geometrical” lifetime of topological cycles.

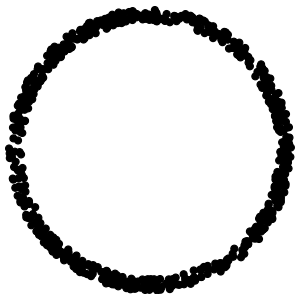
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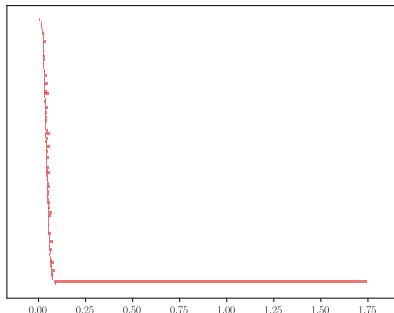
Multiparameter persistence motivation

Motivation: noisy data

An (eyeball-looking) annulus has a *predictable* and *interpretable* topological persistence.



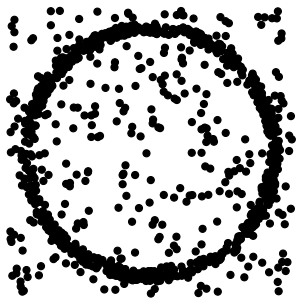
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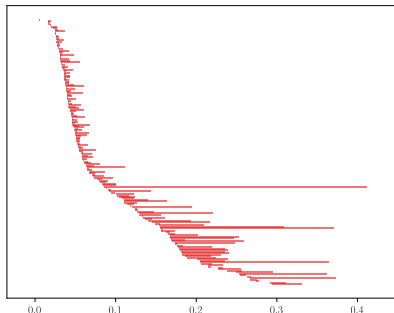
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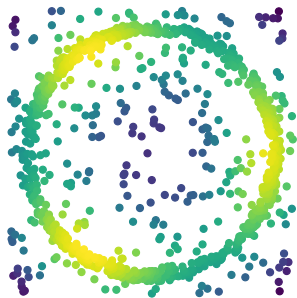
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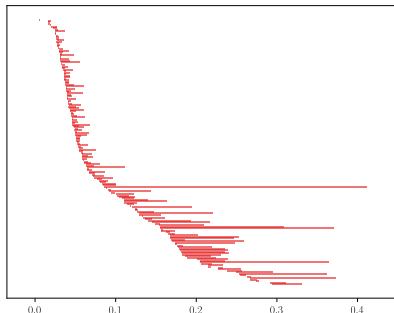
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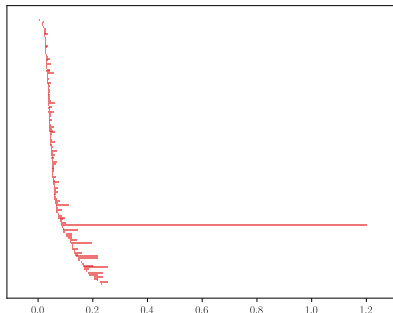
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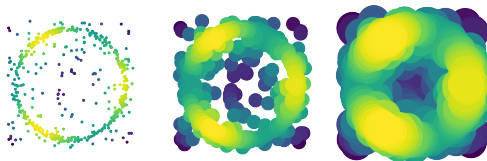


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Q. *Modes* of the measure?

Bipersistence and noise

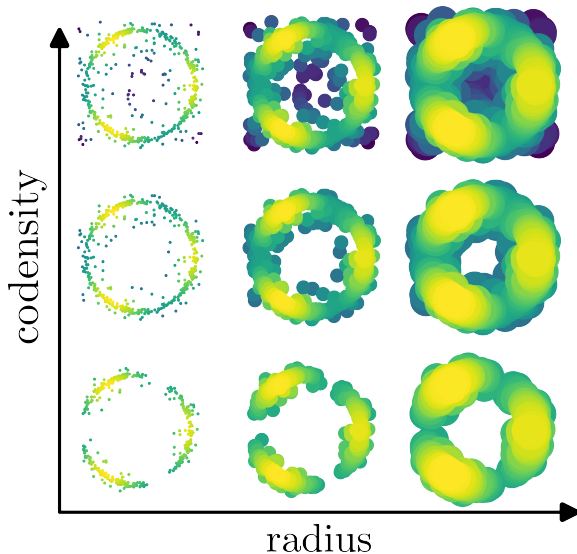
Radius + codensity



radius

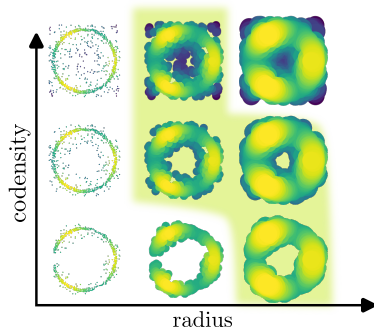
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Goal:

Identify *lifetimes* of topological structures from the multifiltration

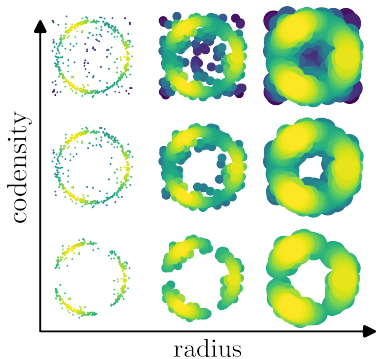
$$F_{(r,d)} = \bigcup_{\text{density}(x) \geq d} B(x, r),$$

i.e., find a *lifetime shape* decomposition of

$$H_i(F, \subseteq).$$

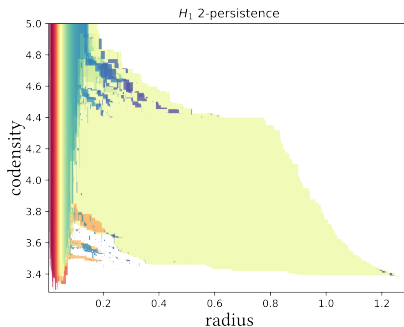
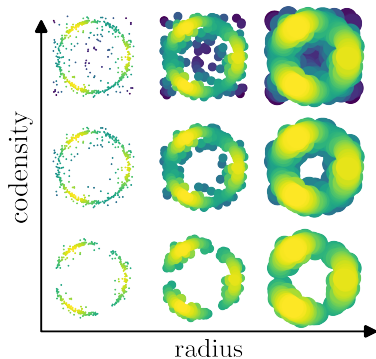
Aside for interpretability: Module approximations

[L., Carrière, Blumberg, 2025]



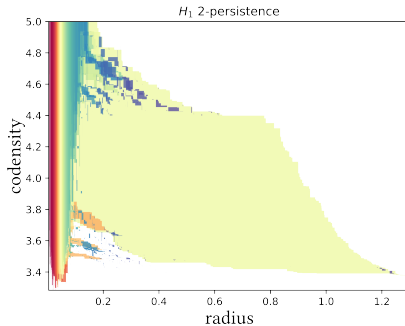
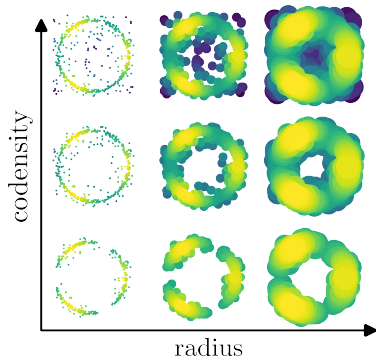
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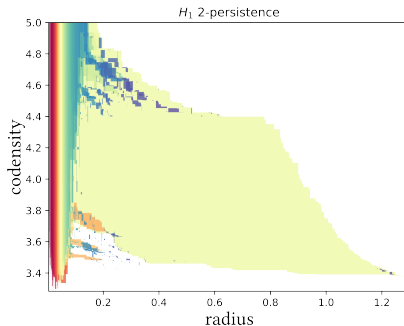
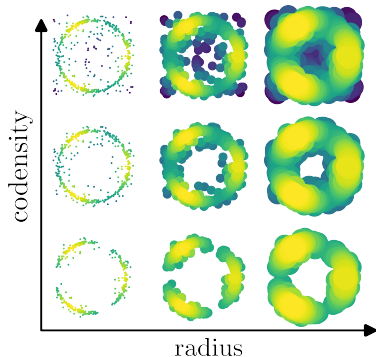
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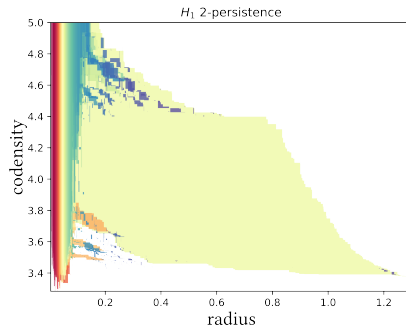
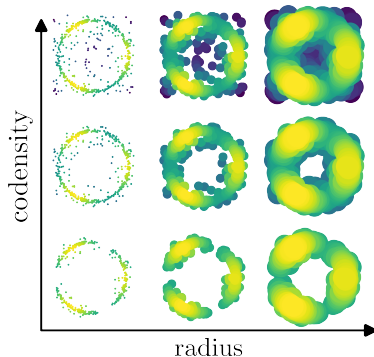


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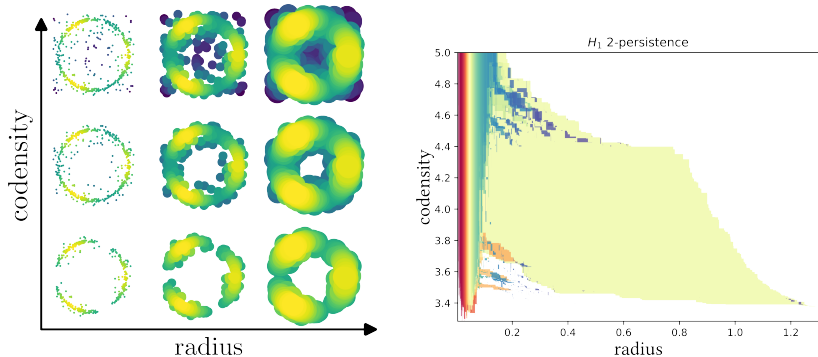


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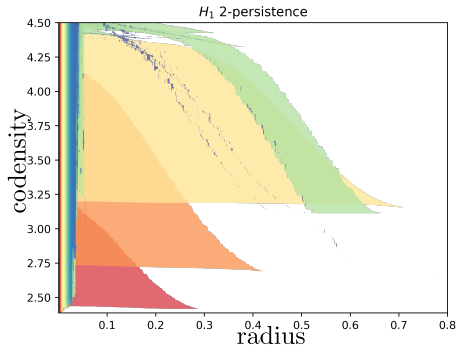
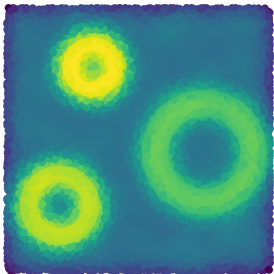
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3. *Large green shape*: annulus' lifetime.

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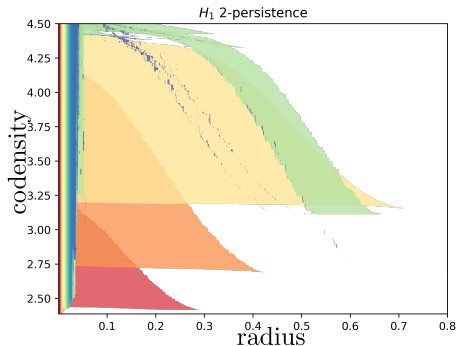
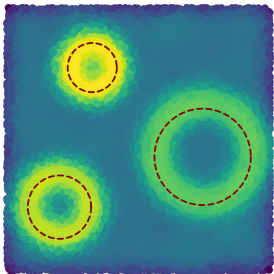
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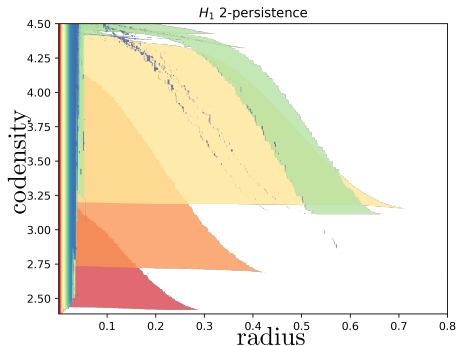
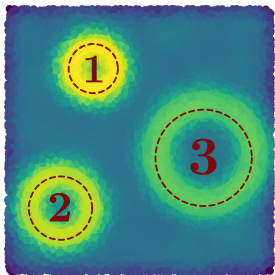
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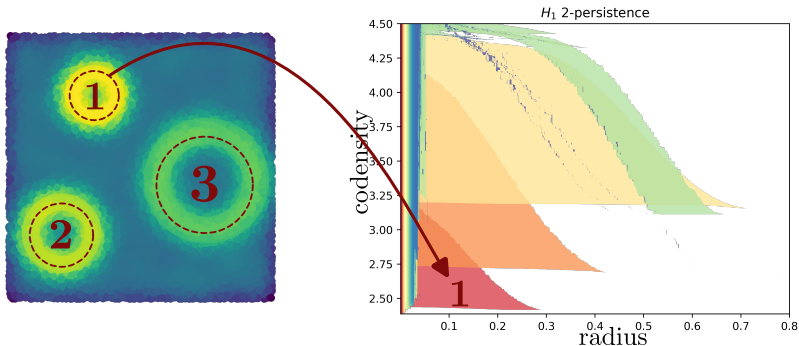
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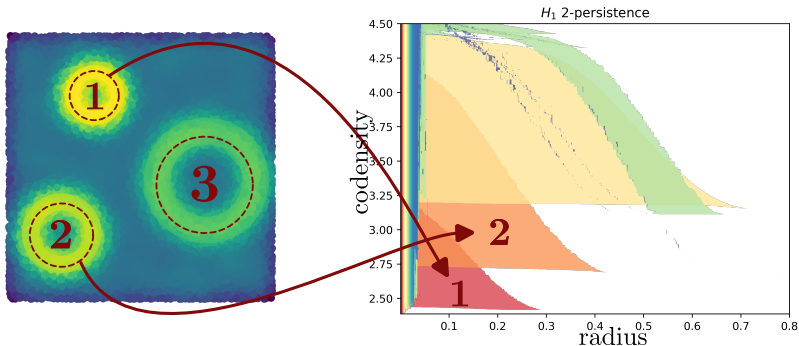
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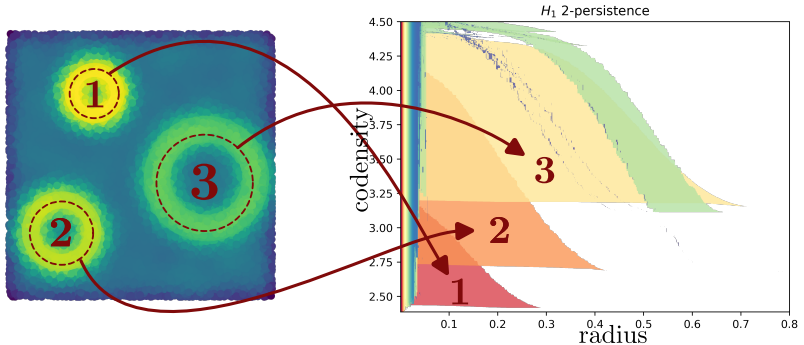
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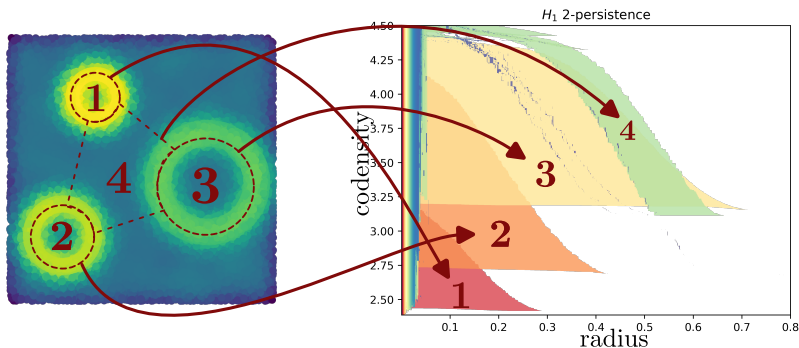
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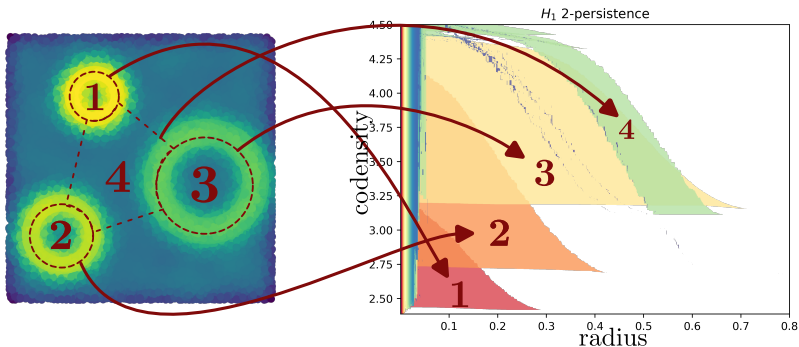
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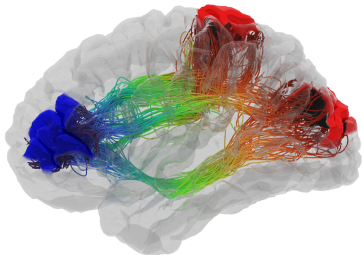
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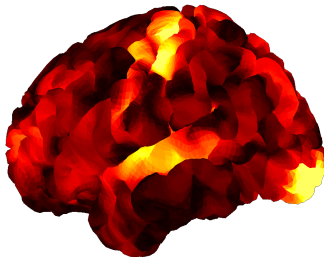


TDA and point clouds

Structural vs Functional MRIs



(a) Structural MRI

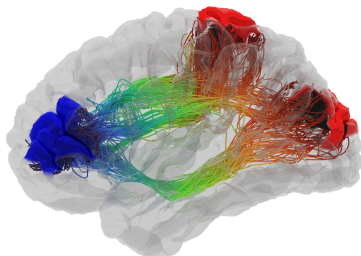


(b) Functional MRI

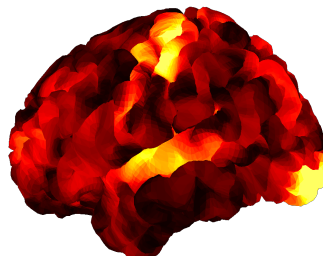
¹Images from the Cronos Inria team.

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$$F_{r,d} := \{\sigma \subseteq \mathcal{P}(\text{Regions}) \mid \text{geodesic diam}(\sigma) \leq r, \min \text{intensity}(\sigma) \geq d\}$$

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Definition (Multifiltration)

A *d-filtration* F is a family of topological spaces $(F_x)_{x \in \mathbb{R}^d}$ with:

$$F_x \subseteq F_y \quad \text{if } x \leq y, \quad \text{where } x \leq y \Leftrightarrow \forall 1 \leq i \leq n, x_i \leq y_i.$$

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If F is an *d-filtration*, then we want to look at $H_i(F, \subseteq)$, where:

$$H_i(F, \subseteq) = \left\{ \begin{array}{l} (H_i(F_x))_{x \in \mathbb{R}^d} \\ \left(H_i(F_x \xrightarrow{t} F_y) : H_i(F_x) \rightarrow H_i(F_y) \right)_{x \leq y \in \mathbb{R}^d} \end{array} \right.$$

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If F is an *d-filtration*, then we want to look at the *multipersistance module* $H_i(F, \subseteq)$, where:

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Definition (Multipersistance module)

An *d-parameter persistence module* is a family of *vector space* $(M_x)_{x \in \mathbb{R}^d}$ with *linear maps* $M(x \leq y) : M_x \rightarrow M_y$ for any $x \leq y \in \mathbb{R}^d$, satisfying:

$$\forall x \leq y \leq z, \quad M(y \leq z) \circ M(x \leq y) = M(x \leq z) \quad \text{and} \quad M(x \leq x) = \text{id}.$$

Discrete approximation

Problem statement.

1. Continuous filtration
2. Discretized filtration

Figure: Suplevelset filtration, inspired from
[Chazal et al., 2011]

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Figure: Filtration by curvature [Song, 2022]

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Geometric-lowerstar multifiltrations

Setting. X compact metric space, $\ell : X \rightarrow \mathbb{R}^d$.

RipsLowerstar (or Function-Rips) multifiltration: [Carlsson, Zomorodian, 2009]

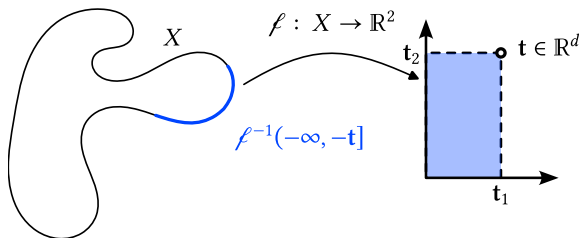
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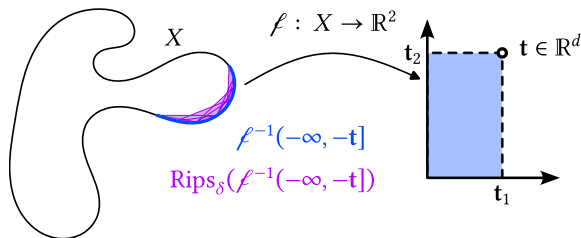


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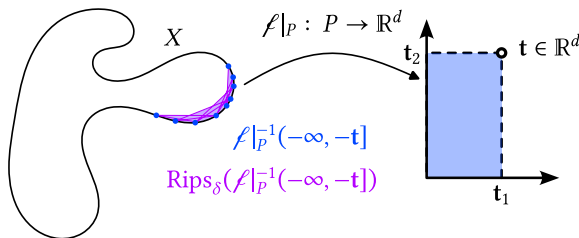


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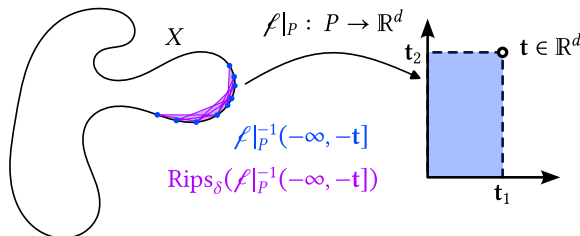


Geometric-lowerstar multifiltrations

Setting. X compact metric space, $\ell : X \rightarrow \mathbb{R}^d$.

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Special case. RipsCodensity.

Geometric-codensity alternatives

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Geometric-codensity alternatives

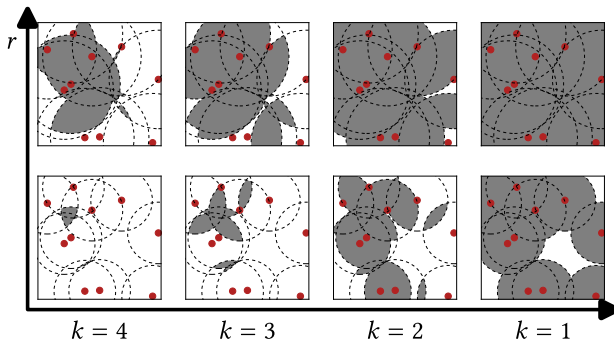
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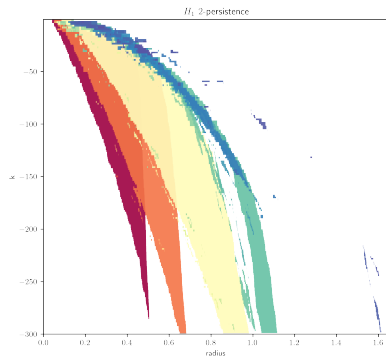
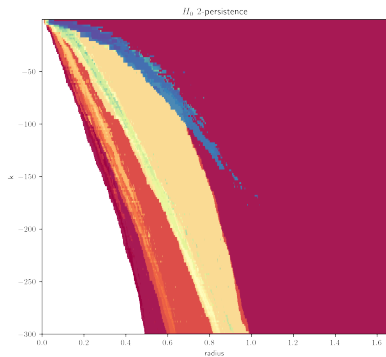
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Multicover example

on the three-annulus dataset



Geometric-codensity alternatives

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Theorem ([Blumberg, Lesnick, 2020] [Rolle, Scoccola, 2020])

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Example

ToMATo [Chazal et al., 2013] or Persistable [Rolle, Scoccola, 2020].

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What about Rips ?

Second attempt

Rips

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Then. [Chazal et al., 2009], [André, Li, L., Oudot, 2025]

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Problem. The parameter $\delta > 0$ depends on the (unknown) sampling error ε .

Third attempt

Lemma ([Cuevas et al., 2004], [Chazal et al., 2015])

If $P = X_n = (x_1, \dots, x_n) \sim \mu^{\otimes n}$, with μ *(a, b)-standard of support X*, i.e.,

$$\forall x \in X, \forall r > 0, \quad \mu(B_X(x, r)) \geq ar^b \wedge 1,$$

then if *X admits an accumulation point*:

$$\left(\frac{1}{an}\right)^{\frac{1}{b}} \lesssim \mathbb{E}[d_H(X_n, X)] \lesssim \left(\frac{\log n}{an}\right)^{\frac{1}{b}}.$$

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This rate is *(quasi-)minimax*.

Sharpness of the bounds

Proof by the example : filtration by the height

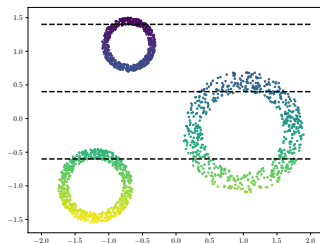


Figure: (left) The dataset.

Sharpness of the bounds

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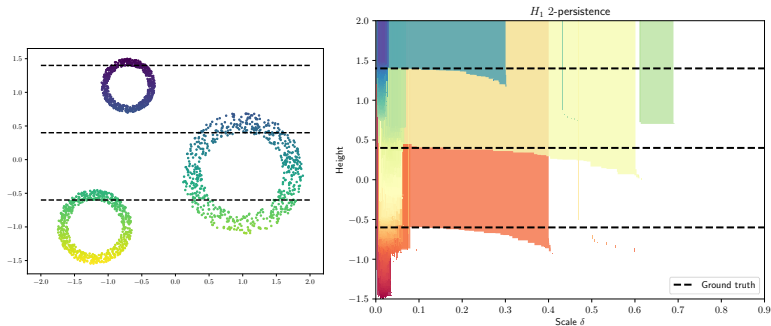


Figure: (left) The dataset. (right) MMA decomposition of the ČechLowerstar filtration.

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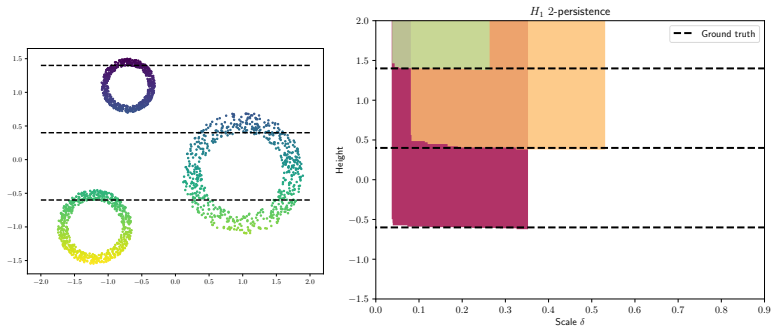


Figure: (left) The dataset. (right) MMA decomposition of the RipsLowerstar-image filtration.

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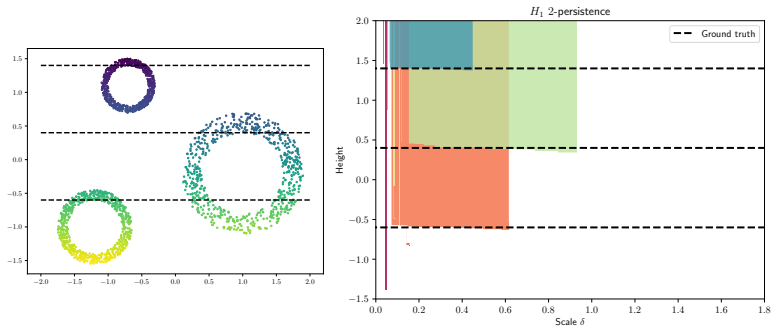


Figure: (left) The dataset. (right) MMA decomposition of *(the approximation of)* the GeodesicRipsLowerstar-image filtration.

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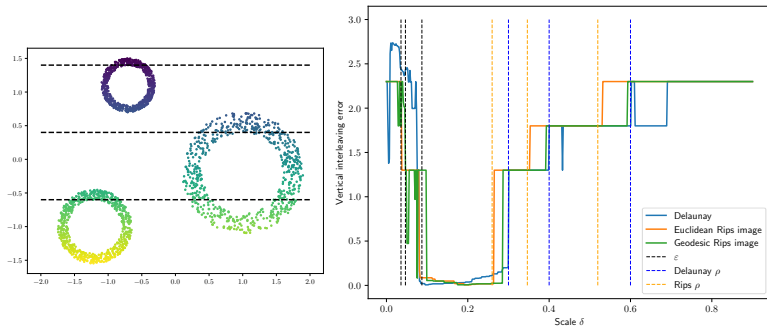


Figure: (left) The dataset. (right) Vertical interleaving error.

Adding noise

Proof by the example : filtration by the height

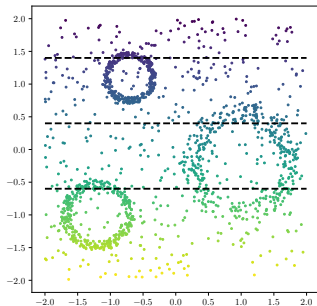


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Adding noise

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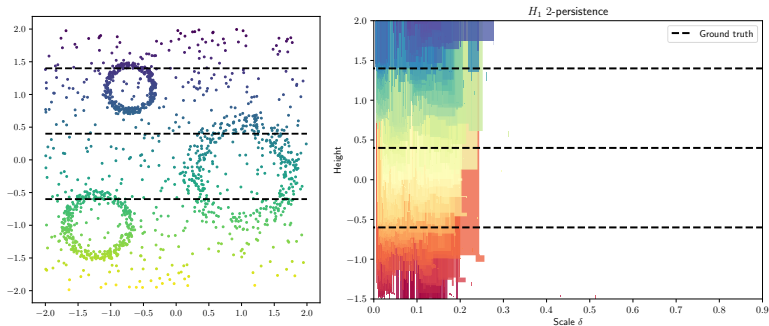


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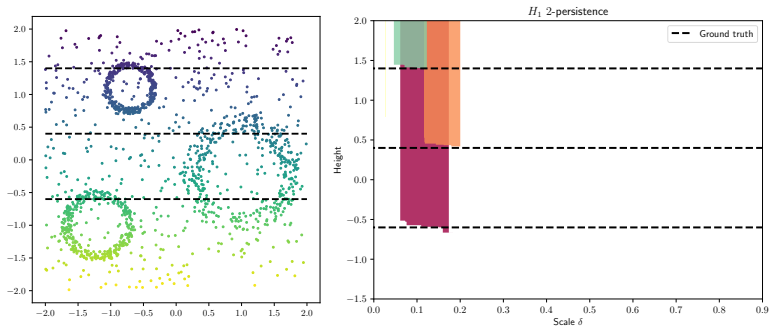


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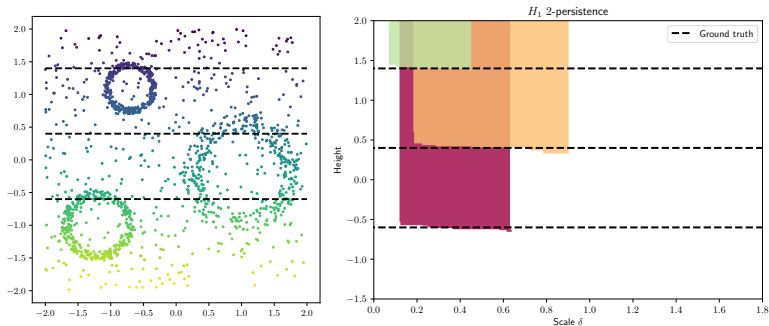


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Adding noise

Proof by the example : filtration by the height

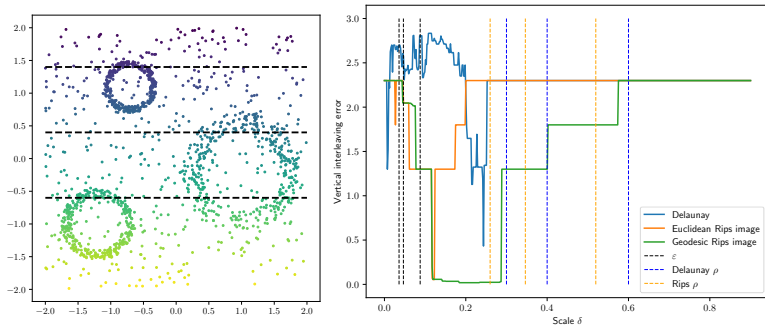


Figure: (left) The dataset. (right) Vertical interleaving error.

Extension to modulus of continuity

Modulus of continuity. $f : X \rightarrow \mathbb{R}^d$ is ω -continuous if

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In particular, α -Hölder functions are controlled by the rate $\left(\frac{\log n}{an} \right)^{\frac{\alpha}{b}}$.

Extension to modulus of continuity

Example: sublevelsets of a brownian motion

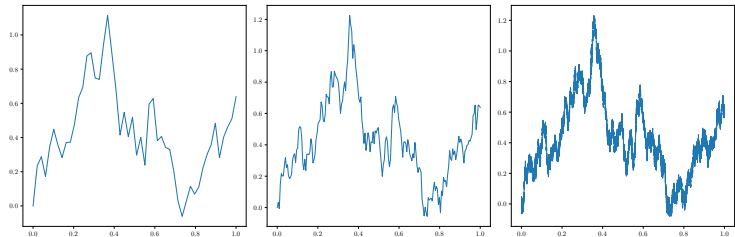


Figure: Iterations that converge to a realization of the standard Brownian motion.

Extension to modulus of continuity

Example: sublevelsets of a brownian motion

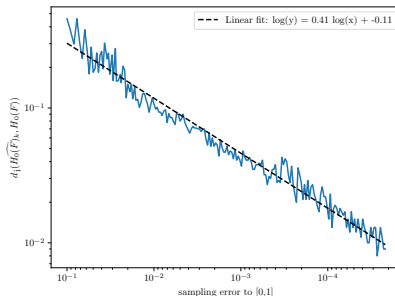
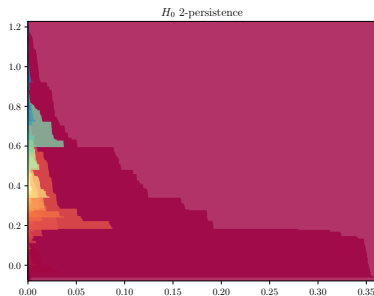
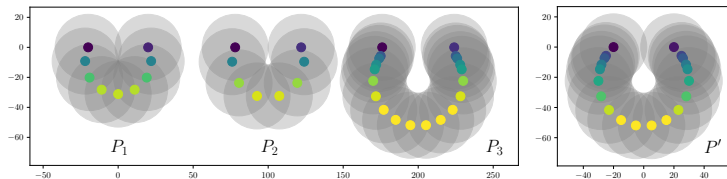


Figure: (left) RipsImageLowerstar=ČechLowerstar MMA of degree 0 (right) Convergence rate of $\widehat{H_*}(\ell)_{\delta_n}$.

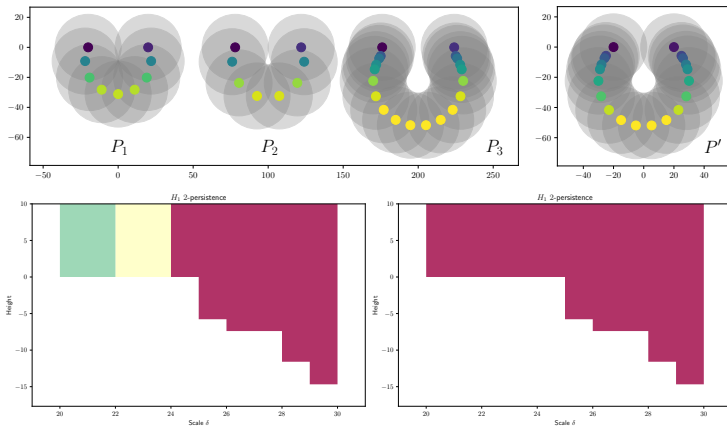
Extension to the multiparameter setting

Vertical slices?



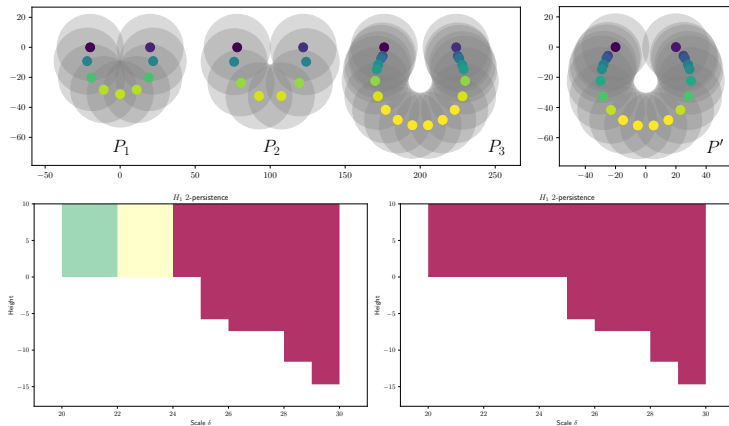
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Extension to the multiparameter setting

Vertical slices?



A similar result holds when slices are replaced with slabs.

Multiparameter convergence

Immunohistochemistry

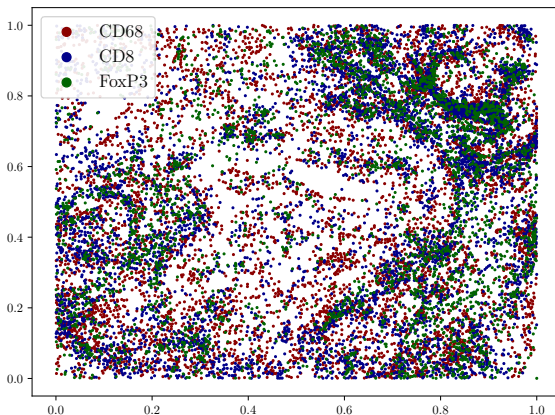


Figure: The dataset, from [Vipond et al., 2021]

Multiparameter convergence

Immunohistochemistry

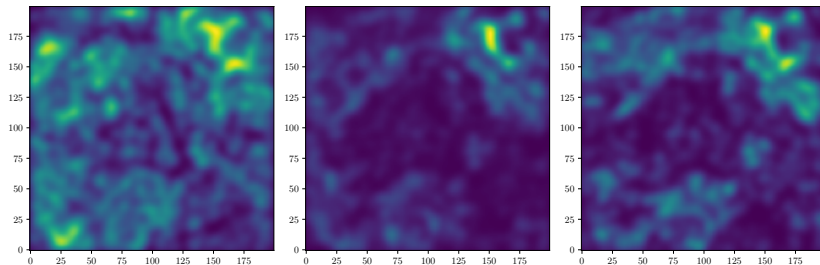


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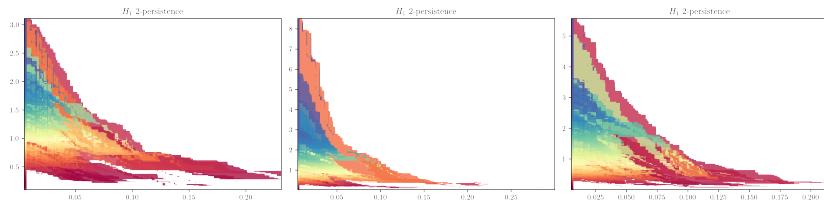
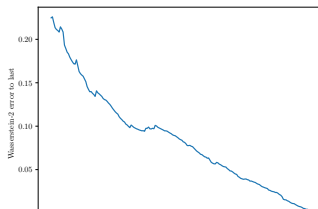
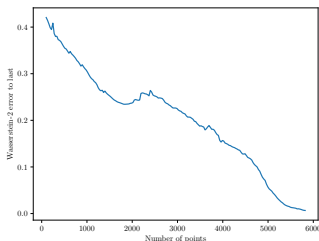
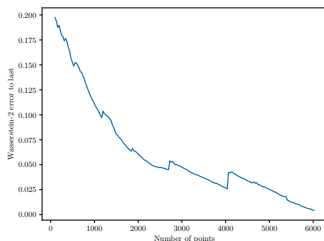


Figure: MMA decompositions of the ČechLowerstar filtrations

Multiparameter convergence

Immunohistochemistry

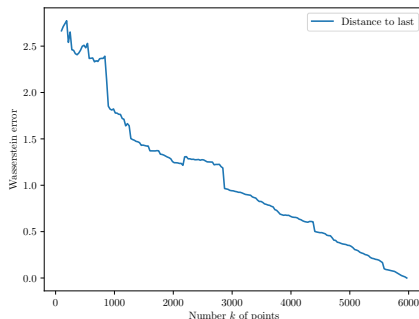
Convergence in 2-param. Wasserstein error (w.r.t. to last) of the rank decomposition of RipsLowerstar on a given protein.



Multiparameter convergence

Immunohistochemistry

Convergence in 4-param. Wasserstein error (w.r.t. to last) of the Hilbert decomposition of RipsImageLowerstar on the three proteins.



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गुडी GUDHI Geometry Understanding
in Higher Dimensions



Multiparameter Persistence
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Thank you!