Optimal sphere packings:
coins and density

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Journées de Géométrie Algorithmique

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Optimal coin packing

Given infinite number of identical coins (),

how to place them on an infinite plane without overlap to maximize the covered surface?

coin packing: 

Optimal coin packing

Given infinite number of identical coins (),

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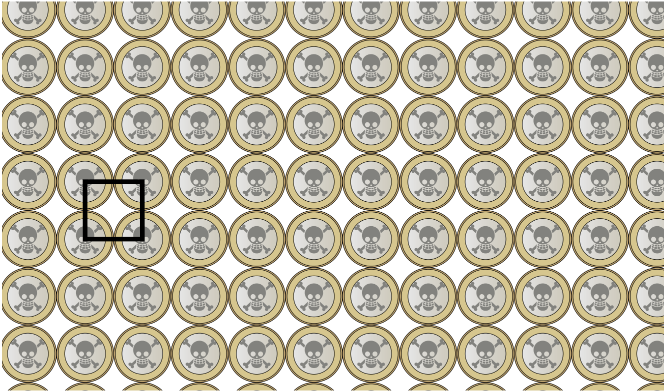
square coin packing:
covers 78% of the surface



Optimal coin packing

Given infinite number of identical coins (),
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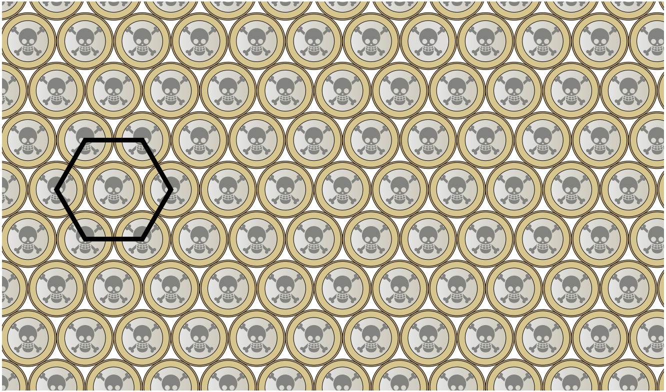
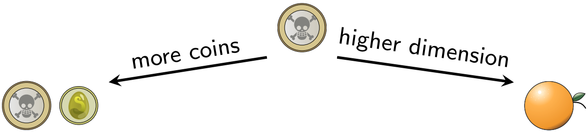
hexagonal coin packing:
covers 90% of the surface

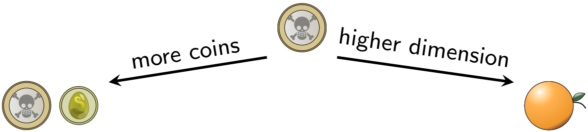


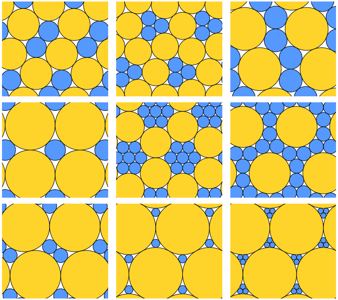
1910–1940

The hexagonal coin packing is optimal.

Introduction







(proved optimal in 2000-2022)

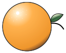
Kepler conjecture, 1611

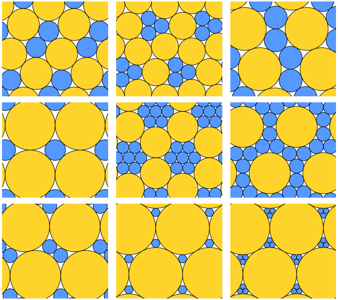
The “cannonball” packing is optimal:

(proved in 1998–2014)

more coins

higher dimension





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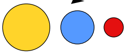

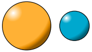

$\mathbb{R}^8, \mathbb{R}^{24}$

(Viazovska, Fields Medal 2022)

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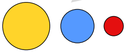
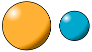




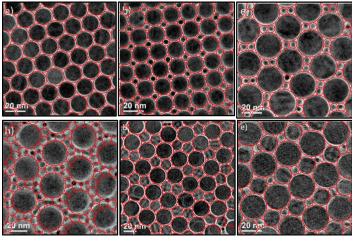
$\mathbb{R}^8, \mathbb{R}^{24}$
(Viazovska, Fields Medal 2022)

Nanomaterials and packings

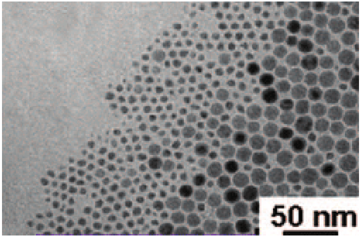
combine different types of nanoparticles
self-assembly

new material

phase separation

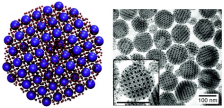


Paik et al 2015



Cheon et al 2006

Also in 3D:

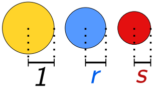


Wu, Fan, Yin 2022

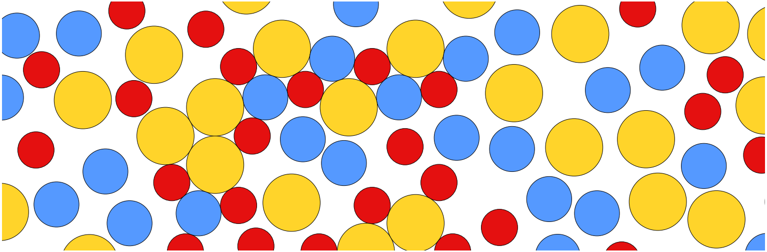
Chemists' question : **which sizes and concentrations allow for new materials?**

Definitions

Discs:



Packing P:
(in $\mathbb{R}^2$)



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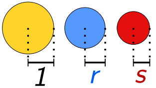

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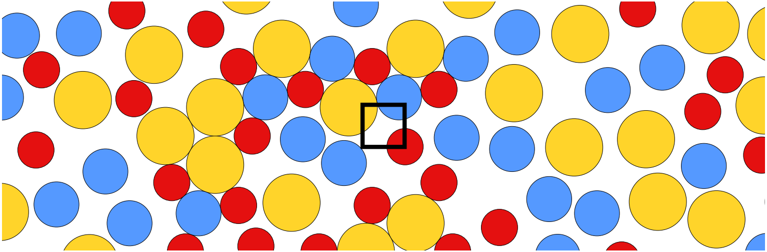
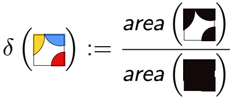


Density:



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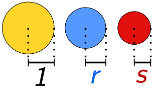

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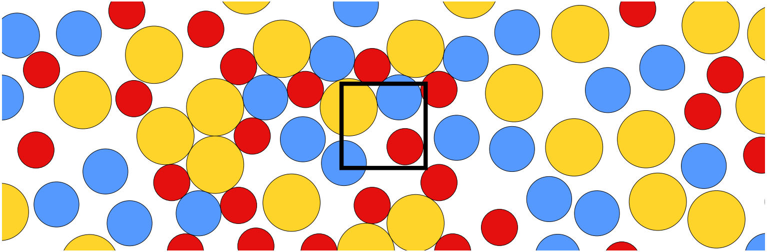

Density:

$$\delta\left(\overset{n}{\underset{n}{\blacksquare}} \cap P\right) := \frac{\mathit{area}\left(\overset{n}{\underset{n}{\blacksquare}} \cap P\right)}{\mathit{area}\left(\overset{n}{\underset{n}{\blacksquare}}\right)}$$

Definitions

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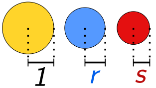

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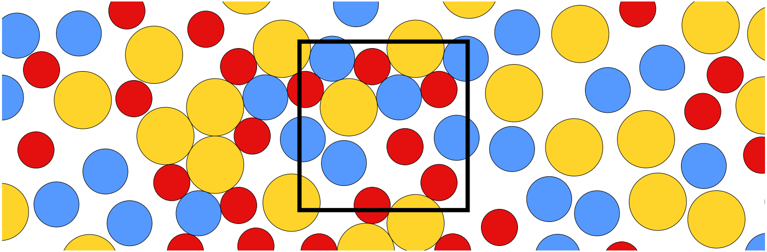

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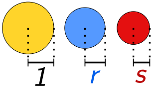

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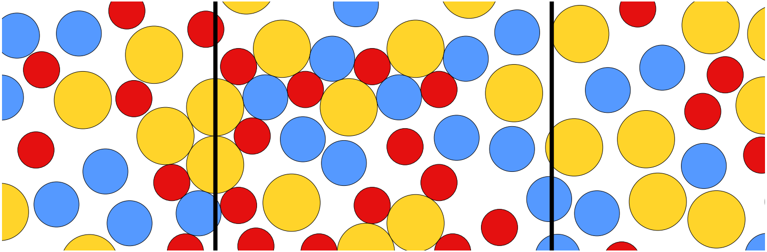

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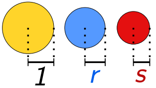

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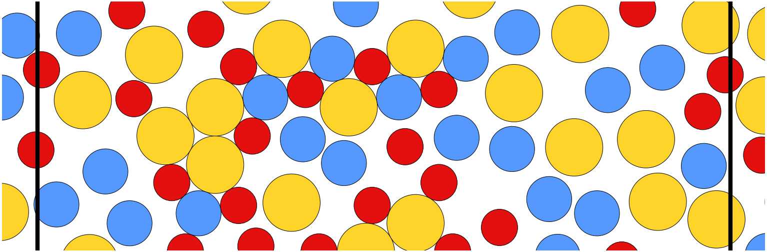

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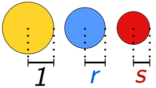

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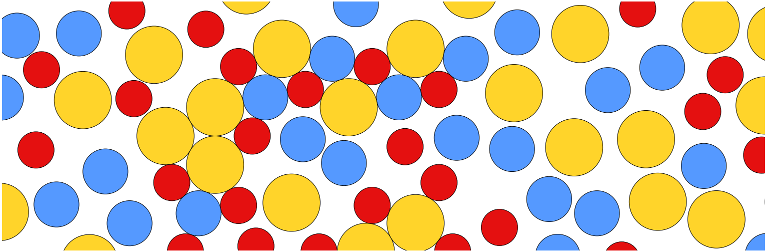

Density:

$$\delta\left(P\right) := \limsup_{n\to\infty} \frac{\mathit{area}\left(n \updownarrow \overset{\overset{n}{\leftrightarrow}}{\blacksquare} \cap P \right)}{\mathit{area}\left(n \updownarrow \overset{\overset{n}{\leftrightarrow}}{\blacksquare} \right)}$$

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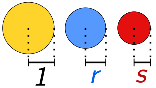

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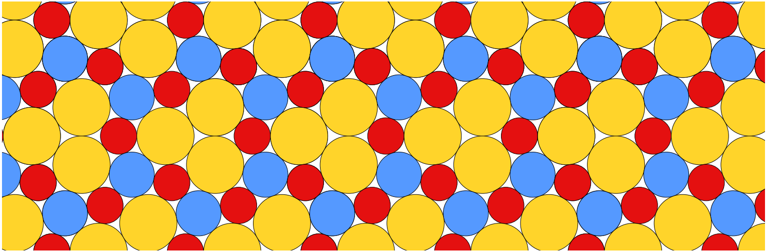


Density:

$$\delta^* \approx 90.9\%$$

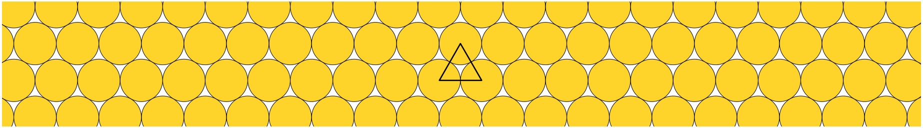
Main Question

Given a finite set of discs (e.g., ),
what is the maximal density δ^* of a packing?

$$\delta^* := \max_P \delta(P)$$

Triangulated packings

A packing is called **triangulated** if its contact graph is a triangulation:



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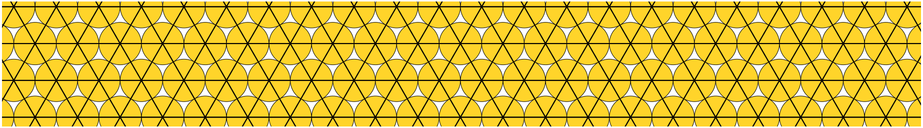

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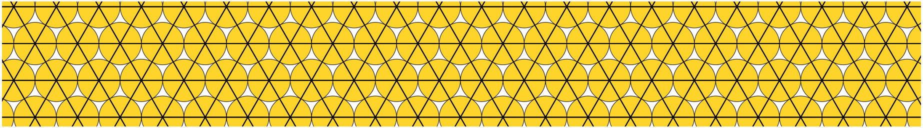
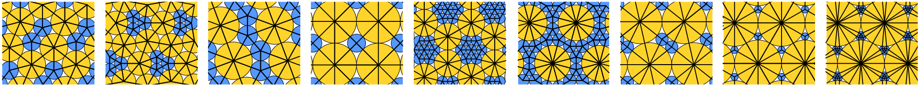


Kennedy, 2006

(Packings by discs of radii 1 and r) There are 9 values of r allowing triangulated packings:



$r \approx 0.63$ $\delta^* \approx 91.1\%$ | $r \approx 0.54$ $\delta^* \approx 91.1\%$ | $r \approx 0.53$ $\delta^* \approx 91.4\%$ | $r \approx 0.41$ $\delta^* \approx 92\%$ | $r \approx 0.38$ $\delta^* \approx 92\%$ | $r \approx 0.34$ $\delta^* \approx 92.5\%$ | $r \approx 0.28$ $\delta^* \approx 93.2\%$ | $r \approx 0.15$ $\delta^* \approx 95\%$ | $r \approx 0.1$ $\delta^* \approx 96\%$

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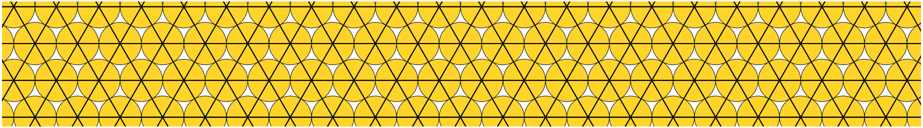

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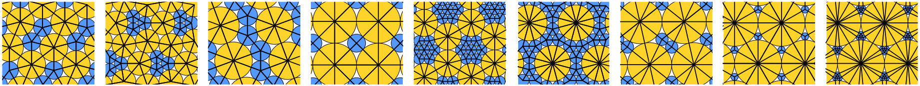

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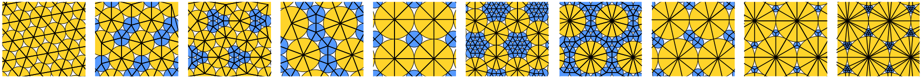
Theorem (Heppes 2000, 2003, Kennedy 2005, Bedaride and Fernique 2022)

Each of these packings is optimal (densest) for discs of radii 1 and r.

Triangulated = optimal?

Optimal triangulated packings:



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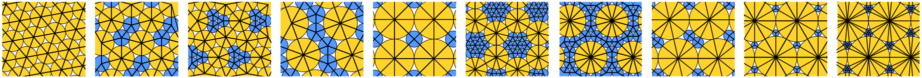
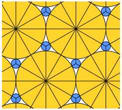
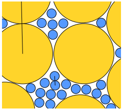


Conjecture (Connelly 2018)

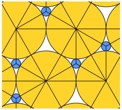
If a finite set of discs allows **saturated** triangulated packings then one of them is optimal.



triangulated
saturated



non triangulated
saturated



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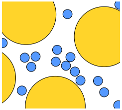

non triangulated
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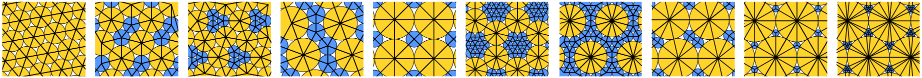

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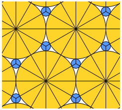

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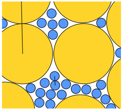

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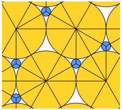

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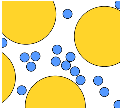
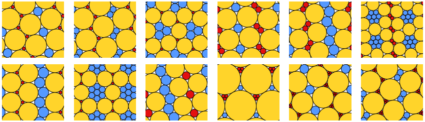


non triangulated
non saturated

Theorem (Fernique, Hashemi, Sizova 2019)

Discs of radii 1, r and s: there are 164 pairs (r, s) allowing triangulated packings.



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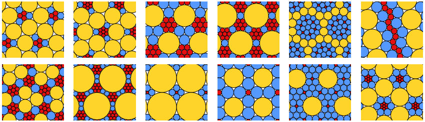
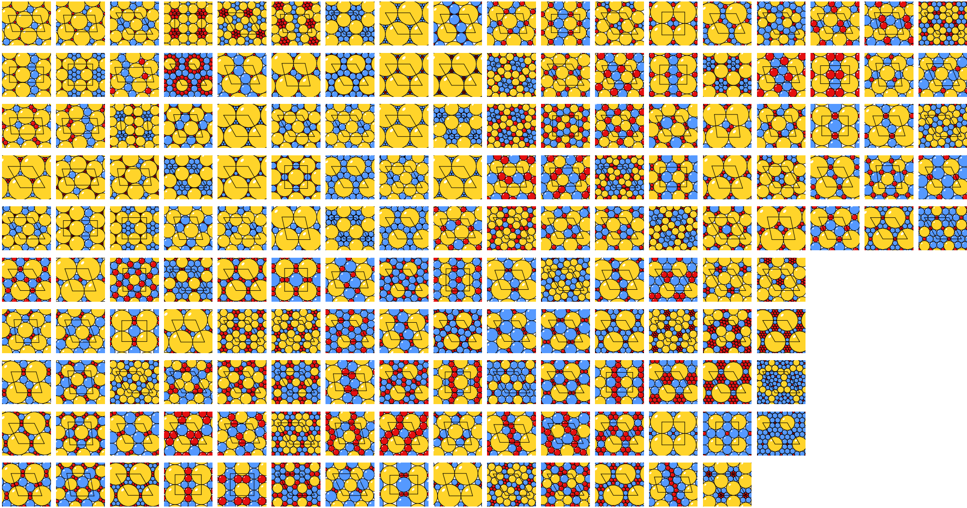
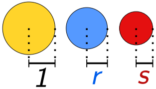
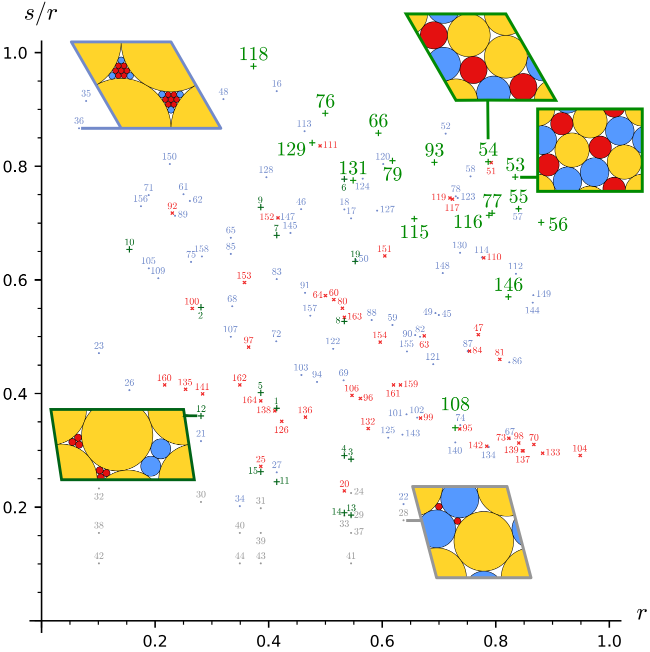






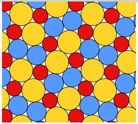
164 (r, s) allowing triangulated packings:

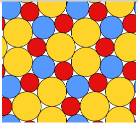
- 15 cases: non saturated
- 16+16 cases: a **ternary** or **binary** triangulated packing is densest
- 45 cases: a binary **non** triangulated packing is denser

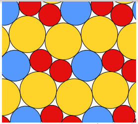


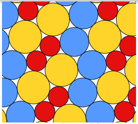
Theorem (Fernique, P 2023)

Each of the following packings is optimal for discs of radii 1, r and s:


$r{\approx}0.8$ $s{\approx}0.7$ $\delta^*{\approx}90.9\%$


$r{\approx}0.8$ $s{\approx}0.6$ $\delta^*{\approx}90.9\%$


$r{\approx}0.8$ $s{\approx}0.6$ $\delta^*{\approx}91\%$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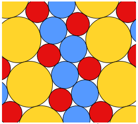
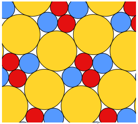
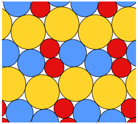
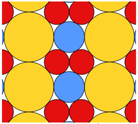
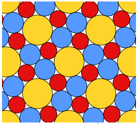
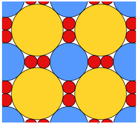

$r{\approx}0.6$ $s{\approx}0.5$ $\delta^*{\approx}91.2\%$

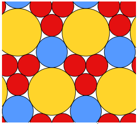

$r{\approx}0.5$ $s{\approx}0.4$ $\delta^*{\approx}91.5\%$


$r{\approx}0.8$ $s{\approx}0.6$ $\delta^*{\approx}91\%$


$r{\approx}0.6$ $s{\approx}0.5$ $\delta^*{\approx}91.3\%$


$r{\approx}0.7$ $s{\approx}0.5$ $\delta^*{\approx}91.1\%$


$r{\approx}0.7$ $s{\approx}0.2$ $\delta^*{\approx}93.2\%$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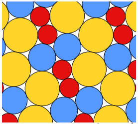
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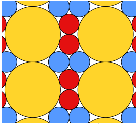
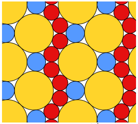
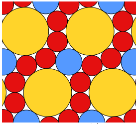
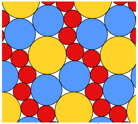
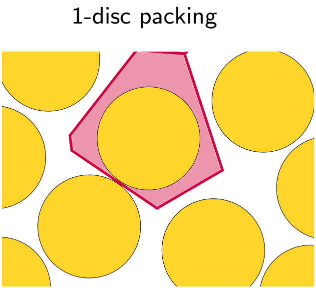

$r{\approx}0.4$ $s{\approx}0.4$ $\delta^*{\approx}92.2\%$

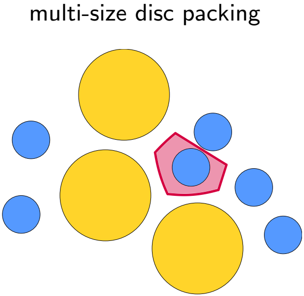

$r{\approx}0.5$ $s{\approx}0.4$ $\delta^*{\approx}91.8\%$


$r{\approx}0.5$ $s{\approx}0.4$ $\delta^*{\approx}91.7\%$


$r{\approx}0.8$ $s{\approx}0.5$ $\delta^*{\approx}91.2\%$

FM-triangulation

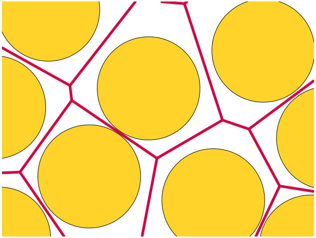




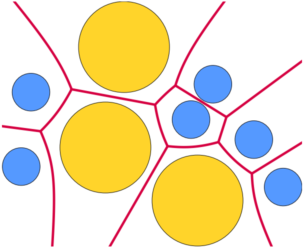
Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

FM-triangulation

1-disc packing



multi-size disc packing



Voronoi cell of a disc in a packing: set of points closer to this disc than to any other

Voronoi diagram of a packing: partition of the plane into Voronoi cells

FM-triangulation

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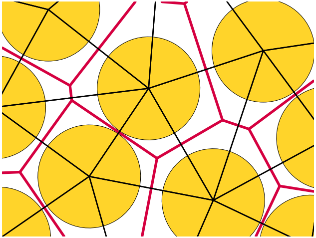

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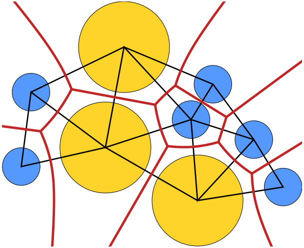

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FM-triangulation of a packing: dual graph of the Voronoi diagram   Fejes Tóth, Mólnar

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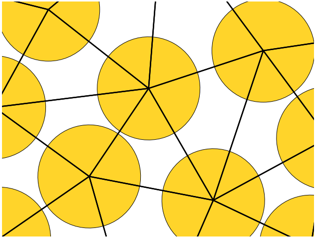

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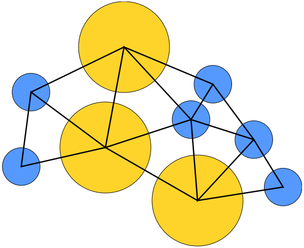

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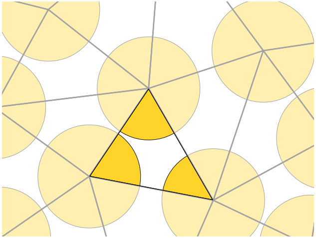

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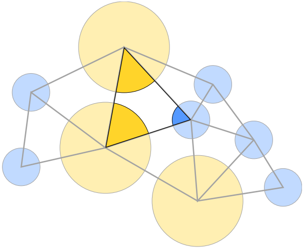

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Density of a triangle Δ in a packing $=$ its proportion covered by discs $\quad \delta_\Delta = \frac{area(\Delta \cap P)}{area(\Delta)}$

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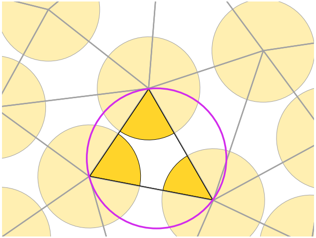

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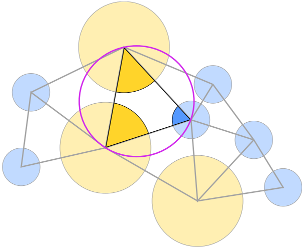

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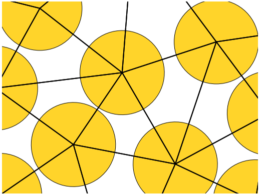
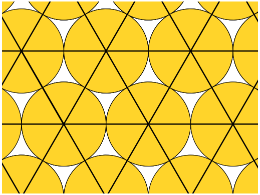
Local density redistribution

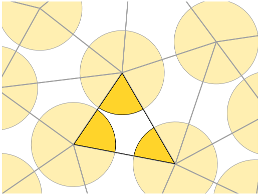


P of density $\delta(P)$



P^* of density δ^*

Local density redistribution



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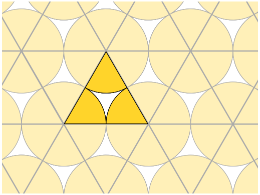
$\forall \Delta,\ \ \delta(\Delta) \leq \delta(\ ) = \delta^*$



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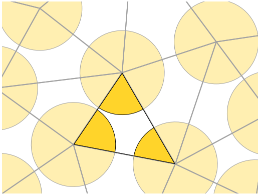

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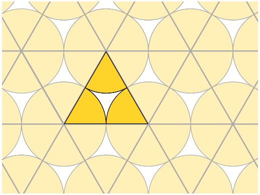

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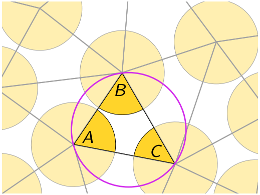

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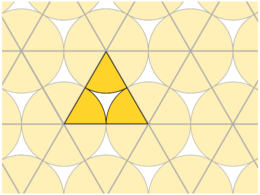

P^* of density δ^*

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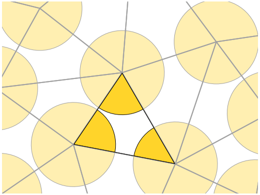
Proof:

- the smallest angle of any Δ is at least $\frac{\pi}{6}$ $\qquad 2 > R = \frac{|AB|}{2\sin\widehat{C}} \geq \frac{1}{\sin\widehat{C}} \Longrightarrow \widehat{C} > \frac{\pi}{6}$
- thus the largest angle is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$
- density of a triangle Δ: $\delta(\Delta) = \frac{\pi/2}{area(\Delta)}$
- the area of a triangle ABC with the largest angle $\hat{A}$: $\frac{|AB|\cdot|AC|\cdot\sin\widehat{A}}{2} \geq \frac{2\cdot2\cdot\frac{\sqrt{3}}{2}}{2} = \sqrt{3}$
- thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}} = \delta^*$

Local density redistribution



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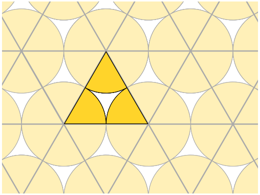

P^* of density δ^*

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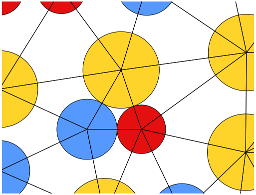
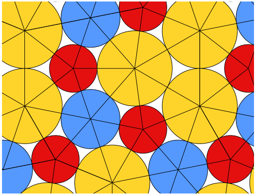


P of density $\delta(P)$



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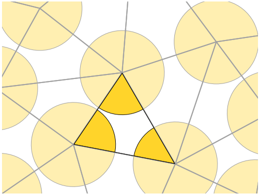

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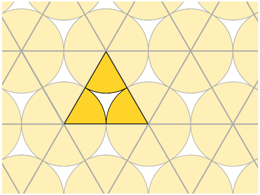

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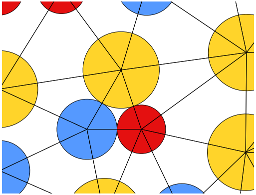
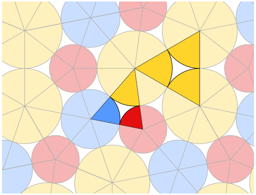


P of density $\delta(P)$

Triangles in P^* have different densities:

$\delta($$) < \delta^* < \delta($$)$

Hopeless to bound the density by δ^* in each triangle...



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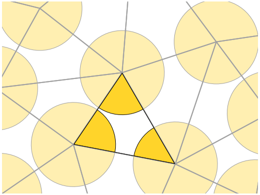

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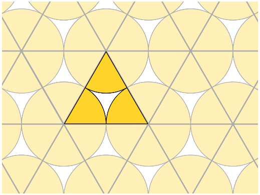

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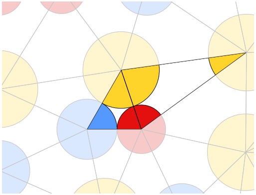
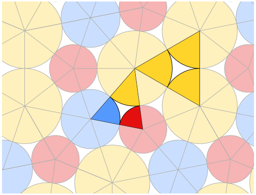


P of density $\delta(P) \leq \delta'(P)$

redistributed density $\delta' \geq \delta$:

dense triangles
share their density
with neighbors



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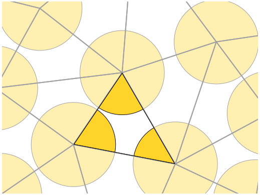

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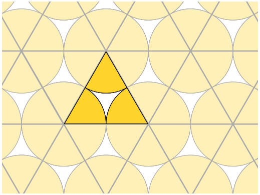

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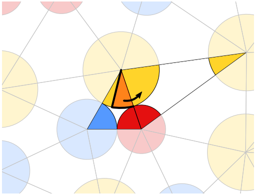

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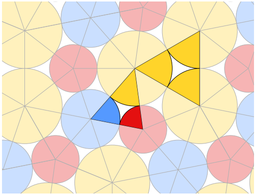

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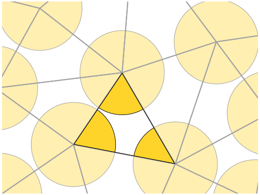

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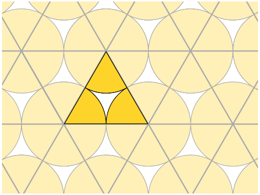

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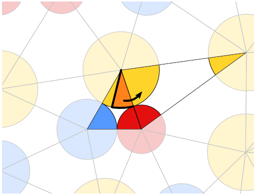

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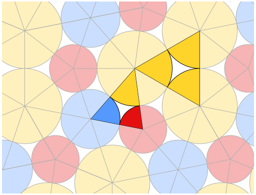

P^* of density δ^*

Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ? (there is a continuum of them)

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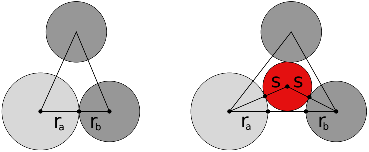
FM-triangulation properties + saturation $\Rightarrow$ uniform bound on edge length



$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

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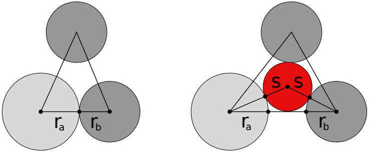
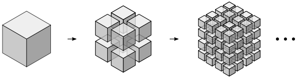


$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

- Interval arithmetic: to verify $\delta'(\Delta_{a,b,c}) \leq \delta^*$ for all $(a,b,c) \in [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] \times [\underline{c}, \overline{c}]$,
  we verify $[\underline{\delta}, \overline{\delta}] \leq \delta^*$ where $[\underline{\delta}, \overline{\delta}] = \delta'(\Delta_{[\underline{a},\overline{a}],[\underline{b},\overline{b}],[\underline{c},\overline{c}]})$

- If $\delta^* \in [\underline{\delta}, \overline{\delta}]$, recursive subdivision:



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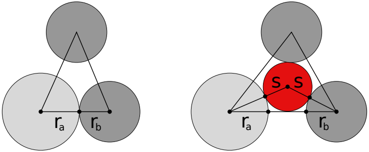

$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

- Interval arithmetic: to verify $\delta'(\Delta_{a,b,c}) \leq \delta^*$ for all $(a, b, c) \in [\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] \times [\underline{c}, \overline{c}]$,

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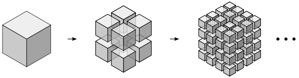
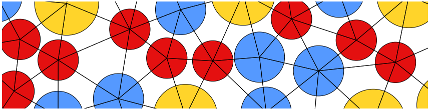
- If $\delta^* \in [\underline{\delta}, \overline{\delta}]$, recursive subdivision:

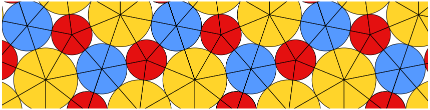


QED

Emptiness instead of density



saturated packing P with the same discs density δ, FM-triangulation $\mathcal{T}$



saturated triangulated packing P^* density δ^*, FM-triangulation $\mathcal{T}^*$

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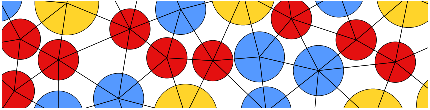

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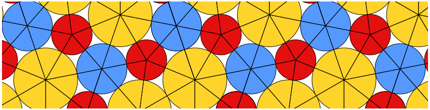

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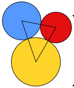
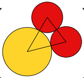
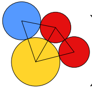
Density function is not additive: $\delta\Big($  $\Big) + \delta\Big($  $\Big) \neq \delta\Big($  $\Big)$ $\longrightarrow$

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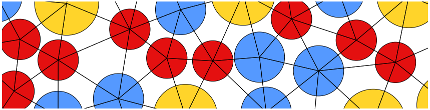

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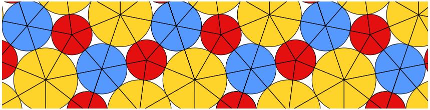

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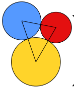
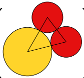
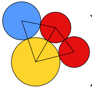
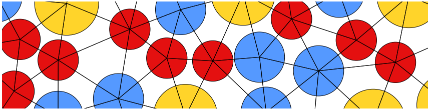
Density function is not additive: $\delta\Bigg($  $\Bigg) + \delta\Bigg($  $\Bigg) \neq \delta\Bigg($  $\Bigg)$ $\longrightarrow$

Emptiness of a triangle $\Delta \in \mathcal{T}$: $E(\Delta) = \delta^* \times area(\Delta) - area(\Delta \cap P)$

$E(\Delta) > 0$ iff the density of Δ is less than δ^*
$E(\Delta) < 0$ iff the density of Δ is greater than δ^*

Additive!

Emptiness instead of density



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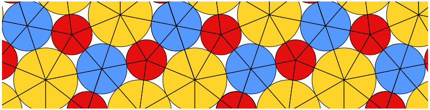

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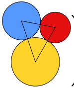
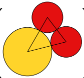
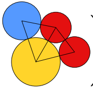
Density function is not additive: $\delta\Big($  $\Big) + \delta\Big($  $\Big) \neq \delta\Big($  $\Big)$ $\longrightarrow$

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Additive!

$$\delta^* \geq \delta \;\Leftrightarrow\; \sum_{\Delta \in \mathcal{T}} E(\Delta) \geq 0$$

Potential is a redistribution of emptiness

We construct a **potential** $U(\Delta) := \overbrace{\dot{U}_\Delta^A + \dot{U}_\Delta^B + \dot{U}_\Delta^C}^{\text{vertices}}$ such that

$\forall$ triangle $\Delta \in \mathcal{T}$, $U(\Delta) \leq E(\Delta)$ (Δ)

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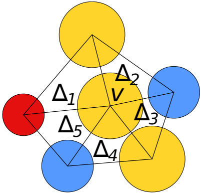

v-corona C_v

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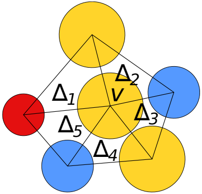

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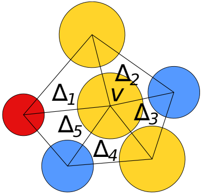

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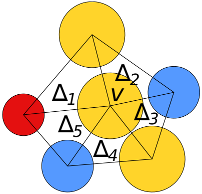

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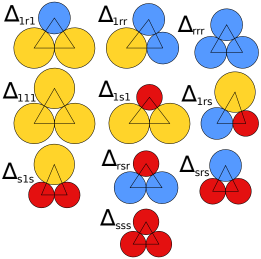
If such U exists then $\delta^* \ge \delta$

Construct it in way that $(\bullet)$ holds and then prove (Δ)

Choosing U to assure $(\bullet)$

Δ_{xyz}	tight triangle: tangent discs of radii x, y, z
$\widehat{xyz}$	angle of Δ_{xyz} in the center of the y-disc
V_{xyz}	potential of Δ_{xyz} in the center of the y-disc



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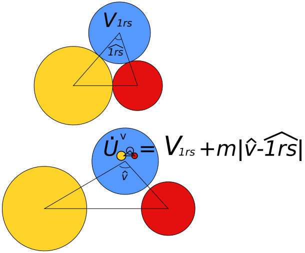

potential of a triangle Δ in v:

$$\dot{U}^{v}_{\Delta} := V_{xyz} + m|\hat{v} - \widehat{xyz}|$$

measures how "far" Δ is from being tight

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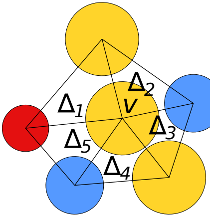

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Choose m to satisfy $\sum_{\Delta \in C_v} \dot{U}_\Delta^v \geq \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C_v}} V_{xyz} + m \times |2\pi - \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C_v}} \widehat{xyz}| \geq 0$ for all coronas C_v

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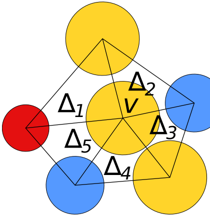

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angle values do not matter $\Rightarrow$ sequence of disc radii $S(C_v)$

FM-triangulation $\Rightarrow$ bounded $|S(C_v)|$

finite number of linear inequalities on m
$\Rightarrow$ computer search

Verifying (Δ) with recursive subdivision

Defining U, we make it as small as possible keeping it positive around any vertrex ($\bullet$)

How to check $U(\Delta) \leq E(\Delta)$ on each possible triangle Δ? (there is a continuum of them)

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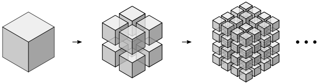
- If $[\underline{U}, \overline{U}]$ and $[\underline{E}, \overline{E}]$ intersect, recursive subdivision: 

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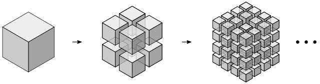

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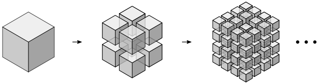
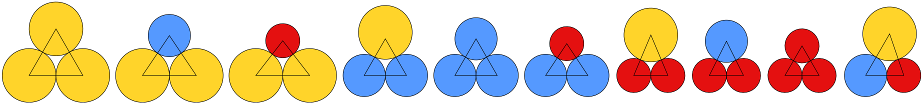
- If $[\underline{U}, \overline{U}]$ and $[\underline{E}, \overline{E}]$ intersect, recursive subdivision: 

never stops if $U(\Delta) = E(\Delta)$

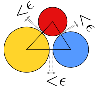
QED ?

Local optima



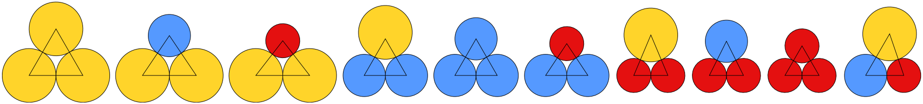
On tight triangles, $U(\Delta_{xyz}) := E(\Delta_{xyz}) \to$ impossible to use interval method around them

ϵ-triangles T_ϵ – triangles close to tight $\Rightarrow$ potential close to emptiness



Under the carpet

Local optima



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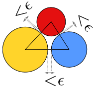

interval arithmetic + recursive subdivision on derivatives on side lengths x_i to check that:

$$\max_{T_\epsilon} \frac{\partial U}{\partial x_i} \Delta x_i < \min_{T_\epsilon} \frac{\partial E}{\partial x_i} \Delta x_i,$$

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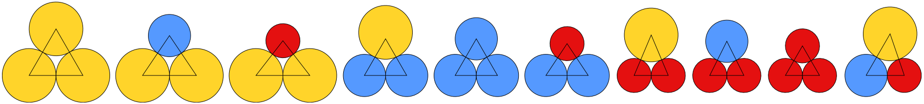

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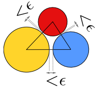

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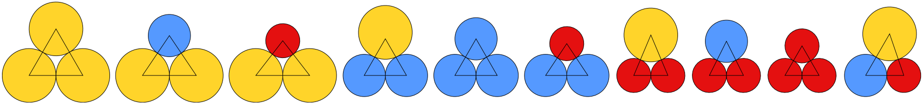
Local optima



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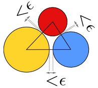

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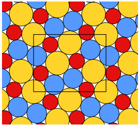
Our proof worked for these cases: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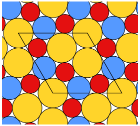
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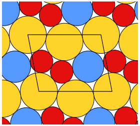
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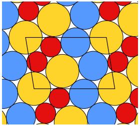
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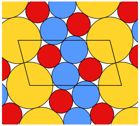
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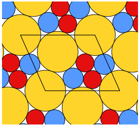
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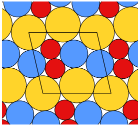
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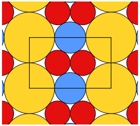
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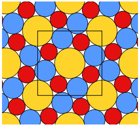
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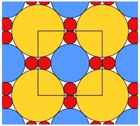
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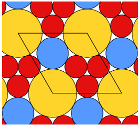
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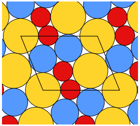
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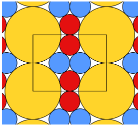
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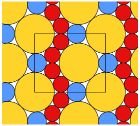
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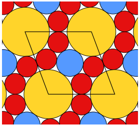
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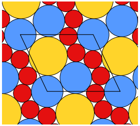
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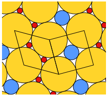
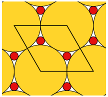
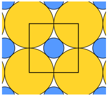
And these:

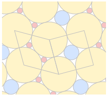
$\delta^* \approx 93\%$ 

$\delta^* \approx 95\%$ 

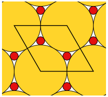
$\delta^* \approx 92\%$ 

And these:

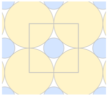
$\delta^* \approx 93\%$



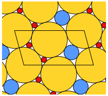
$\delta^* \approx 95\%$



$\delta^* \approx 92\%$



And these: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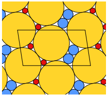
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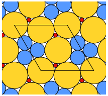
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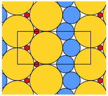
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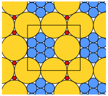
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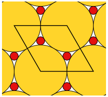
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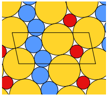
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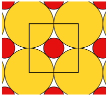
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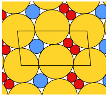
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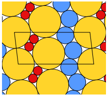
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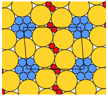
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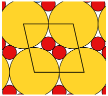
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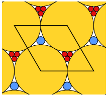
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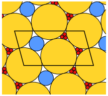
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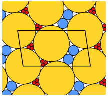
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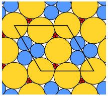
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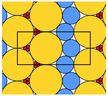
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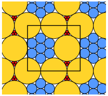
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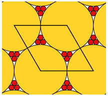
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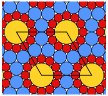
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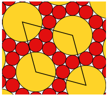
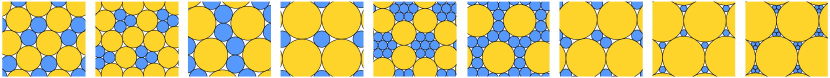
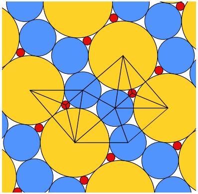

b_6

45 counter examples: *flip-and-flow* method



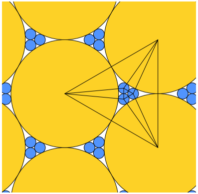
When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

triangulated ternary packing



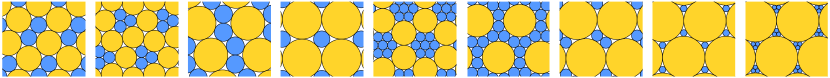
$\delta \leq 0.931369$ $s \approx 0.121445$

dense binary packing



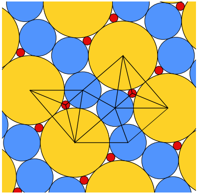
$\delta \approx 0.962430$ $r \approx 0.101021$

45 counter examples: *flip-and-flow* method



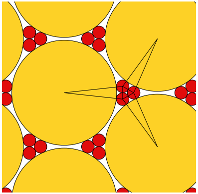
When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

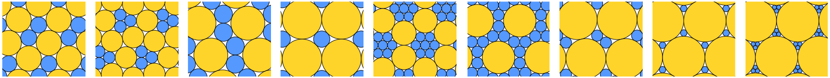
dense non-triangulated packing



$\delta \geq 0.937371$ $s \approx 0.121445$

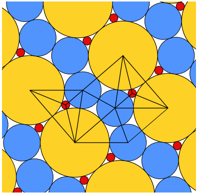
Optimal triangulated packings

45 counter examples: *flip-and-flow* method

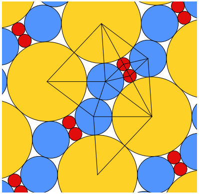


When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

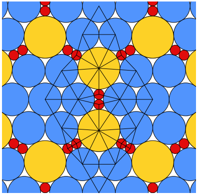
triangulated ternary packing



$\delta \leq 0.931369$ $s \approx 0.121445$

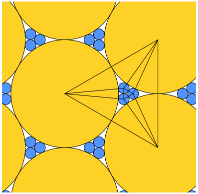


$\delta \leq 0.924522$ $s \approx 0.166169$

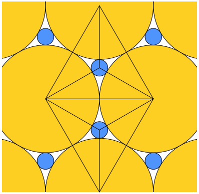


$\delta \leq 0.917352$ $s \approx 0.240205$

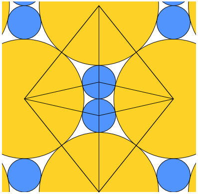
dense binary packing



$\delta \approx 0.962430$ $r \approx 0.101021$

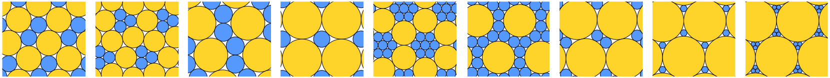


$\delta \approx 0.950308$ $r \approx 0.154701$



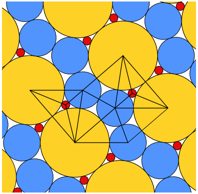
$\delta \approx 0.931901$ $r \approx 0.280776$

45 counter examples: *flip-and-flow* method

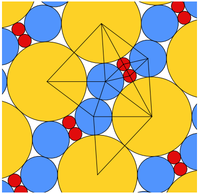


When the ratio of two discs is close enough to the ratio in a dense binary packing, we can pack these discs in a similar (non triangulated) manner and still get high density

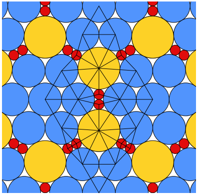
triangulated ternary packing



$\delta \le 0.931369$ $s \approx 0.121445$

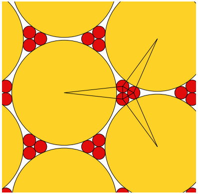


$\delta \le 0.924522$ $s \approx 0.166169$

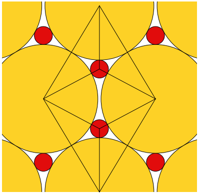


$\delta \le 0.917352$ $s \approx 0.240205$

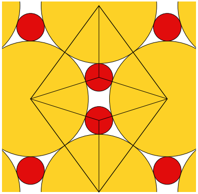
dense non-triangulated packing



$\delta \ge 0.937371$ $s \approx 0.121445$



$\delta \ge 0.939305$ $s \approx 0.166169$

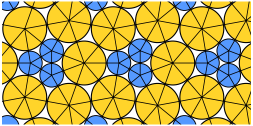


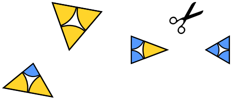
$\delta \ge 0.918420$ $s \approx 0.240205$

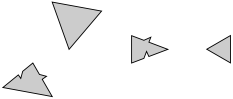
Thank you for your attention!

(Packings and tilings)

triangulated packings $\sim$ tilings by triangles with local rules



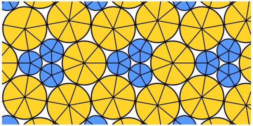


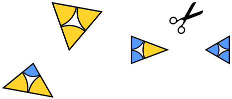


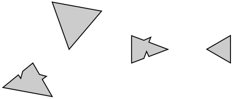
density $=$ weighted proportion of tiles

(Packings and tilings)

triangulated packings $\sim$ tilings by triangles with local rules







density $=$ weighted proportion of tiles

Triangulated Packing Problem

Given k disc radii $\overbrace{r_1, \cdots, r_k}^{\text{algebraic numbers represented by polynomials and intervals}}$, is there a triangulated packing of density $\overbrace{> \frac{\pi}{2\sqrt{3}}}^{\text{excludes hexagonal packing}}$

$\forall\ r_1, \cdots, r_k$ with triangulated packings, one is periodic $\Rightarrow$ **decidable**
(Wang algorithm: search for a period)

$\exists\ r_1, \cdots, r_k$ whose triangulated packings are all aperiodic $\Rightarrow$ **undecidable?**

triangulated packings $\sim$ tilings by triangles with local rules

density = weighted proportion of tiles

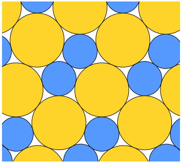
Dense Packing Problem

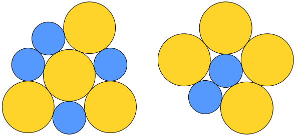
Given k disc radii $\overbrace{r_1, \cdots, r_k}^{\text{algebraic numbers represented by polynomials and intervals}}$, is there a packing of density $> \overbrace{\frac{\pi}{2\sqrt{3}}}^{\text{excludes hexagonal packing}}$

$\forall\ r_1, \cdots, r_k$ with dense packings, one is periodic $\Rightarrow$ **decidable**
(interval arithmetic and subdivision until needed precision)

$\exists\ r_1, \cdots, r_k$ whose dense packings are all aperiodic $\Rightarrow$ **not possible!**

packing is triangulated $\iff$ each disc has a "corona"





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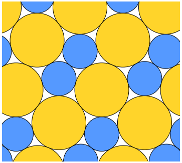
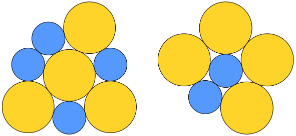
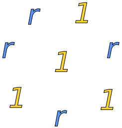




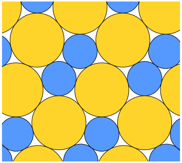
To find disc sizes with triangulated packings, we run trough all possible combinations of symbolic coronas of two discs (finite number):

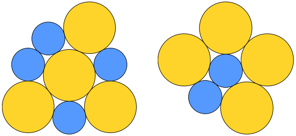
symbolic corona



(Fernique, Hashemi, Sizova 2019)

packing is triangulated $\Longleftrightarrow$ each disc has a "corona"





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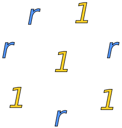
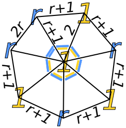
symbolic corona $\longrightarrow$ value of r





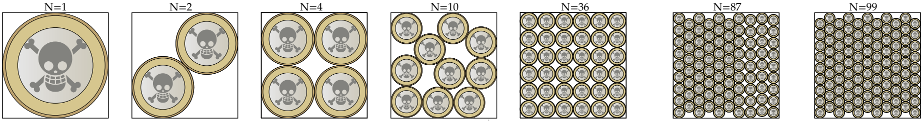
$$6 \times \widehat{11r} + 1 \times \widehat{r1r} = 2\pi$$

$$r \approx 0.63$$

(Fernique, Hashemi, Sizova 2019)

Coins in squares and circles

Place N identical coins in a smallest possible square   solved for 1–30, 36 (1964–2005)



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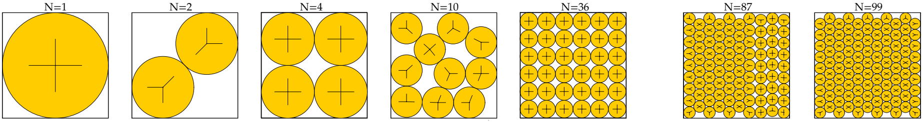
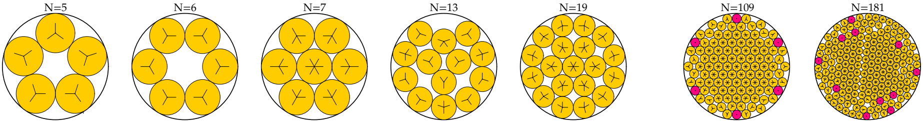


. . . circle solved for 1–13, 19 (1967–2003)

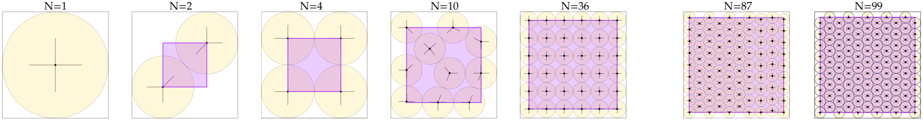


Coins in squares and circles

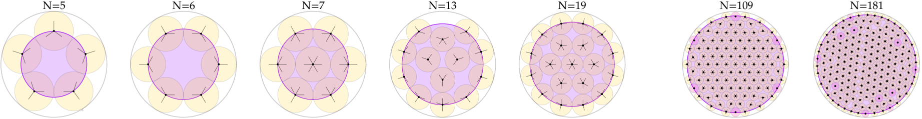
Maximize $\min_{1 \leq i < j \leq N} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

Place N identical coins in a smallest possible square solved for 1–30, 36 (1964–2005)

$0 \leq x_i, x_j, y_i, y_j \leq 1$



$0 \leq x_i^2 + y_i^2,\ x_j^2 + y_j^2 \leq 1$ . . . circle solved for 1–13, 19 (1967–2003)



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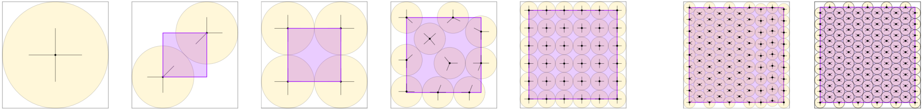
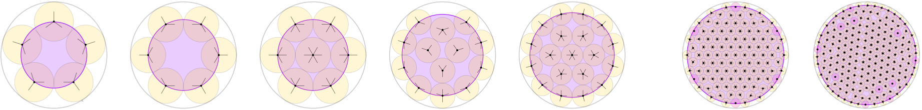
N=1 N=2 N=4 N=10 N=36 N=87 N=99



$0 \le x_i^2 + y_i^2,\ x_j^2 + y_j^2 \le 1$ ... circle solved for 1–13, 19 (1967–2003)

N=5 N=6 N=7 N=13 N=19 N=109 N=181



Find candidates: by hand for small N, billiard simulation, perturbation method

Prove optimality: by hand for small N and circles, interval analisys, branch-and-bound