Optimal sphere packings:
from coins to oranges

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Journées de Géométrie Algorithmique

14 Oct 2025

1. Sphere packings in 3D
2. Density bounds: from discs to spheres
3. Optimality proofs for
4. Conclusion

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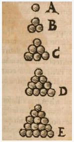
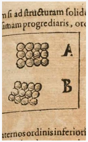
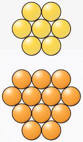
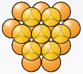
Kepler conjecture: -packings

3D close -packings:









$$\delta^* = \frac{\pi}{3\sqrt{2}} \approx 74\%$$

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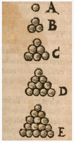

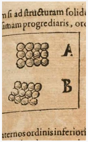

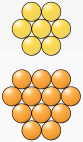

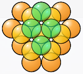

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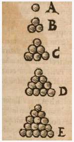

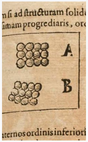

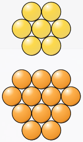

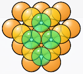

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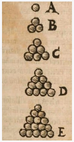

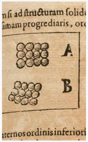

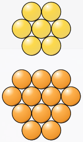

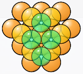

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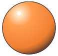
Hales, Ferguson, 1998–2014 (Conjectured by Kepler, 1611)

Close packings are optimal.

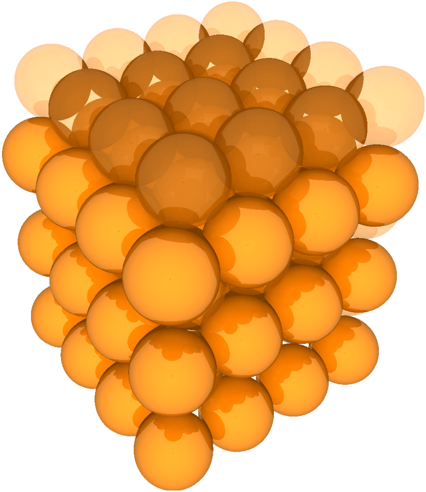
- close packings are optimal among lattice packings — Gauss, 1831
- 18th problem of the Hilbert's list — 1900
- 6 preprints by Hales and Ferguson — ArXiv 1998
  250 pages and $>$ 180000 lines of code
- reviewing: 13 reviewers, 4 years... "99% certain" — 1999–2003
- published proof: 300 pages, 3 computer programs — DCG 2006
- Flyspeck project: formal proof (HOL Light proof assistant) — 2003–2014
  Forum of Mathematics, Pi 2017

Rock salt -packings

sphere



close packing



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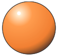

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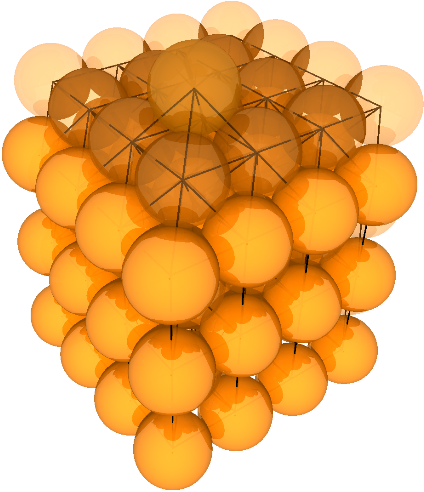
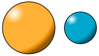


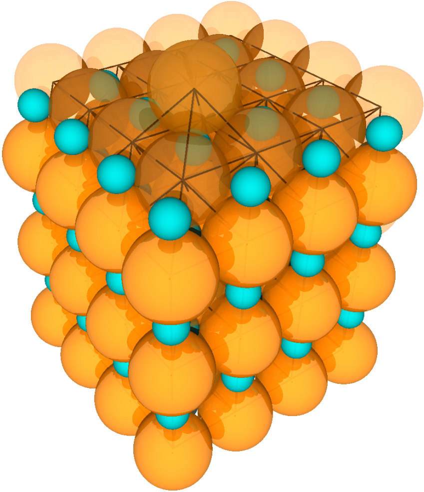
Rock salt -packings

rock salt spheres



1 $r=\sqrt{2}-1$

rock salt packing



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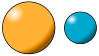

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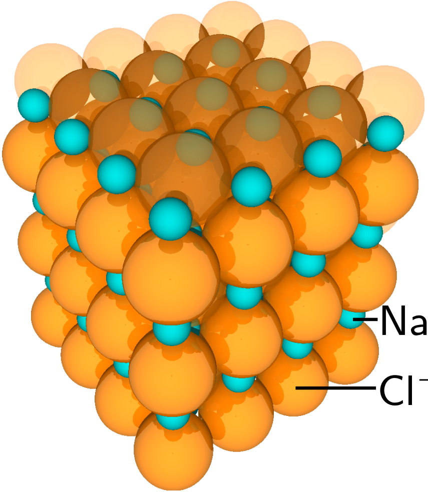

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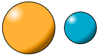


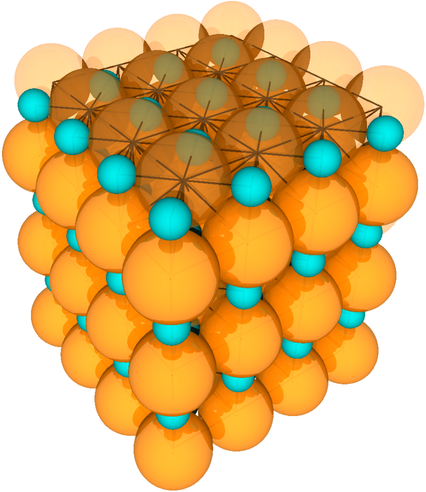
1 $\quad r = \sqrt{2} - 1$

triangulated $\rightarrow$ simplicial
(contact graph is a "tetrahedration")

Fernique, 2019

The only simplicial 2-sphere packings in 3D are rock salt packings.

rock salt packing



Rock salt -packings

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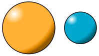

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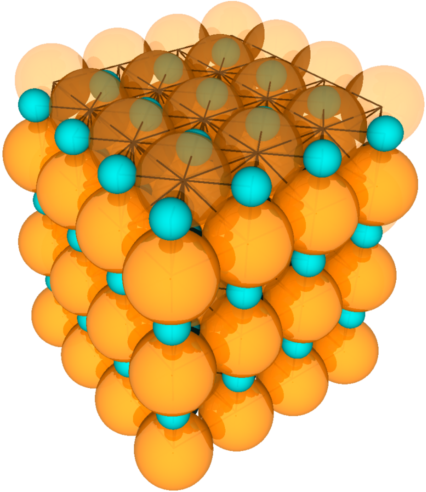
Salt conjecture **open problem**

Rock salt packings are optimal
$\delta^* = (\frac{5}{3} - \sqrt{2})\pi \approx 79\%$.

rock salt packing



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Uniformity and Florian bound

Packing of **uniformity** q: packing by discs of radii $\in [q, 1]$.

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Florian, 1960

The density of a packing of uniformity q never exceeds $\delta_F(q) := \delta\Bigg($ 

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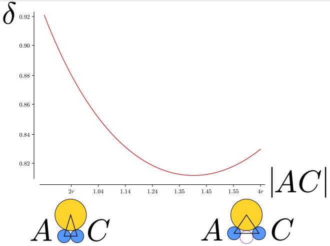
Proof:

- Among all triangles with 2 contacts between discs,  is the densest.



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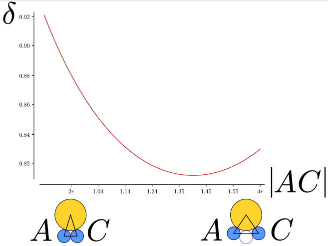
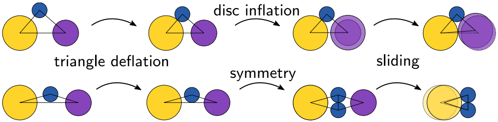
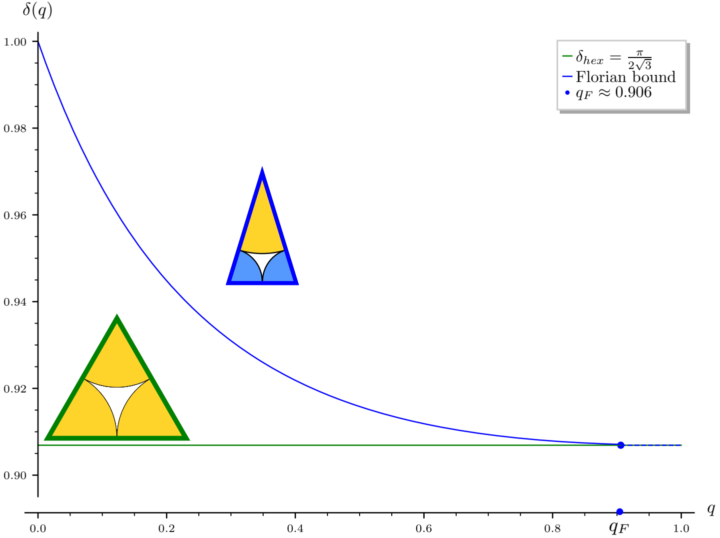


- For any triangle, there is a denser triangle with at least two contacts.

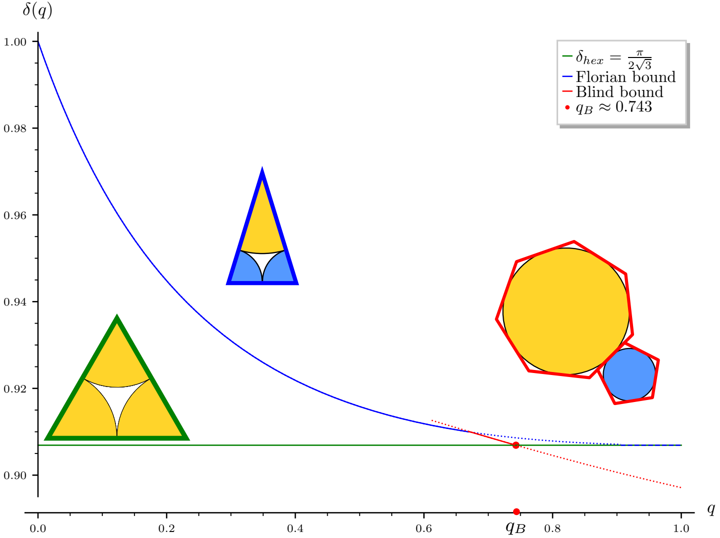
Reduce the dimension of the set of triangles ($3 \to 1$) Fejes Tóth, Mólnar, 1958



Florian bound 1960

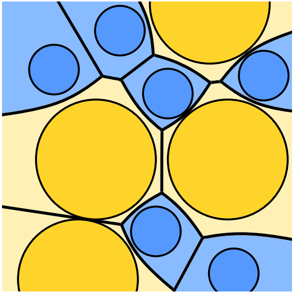


Blind bound 1969



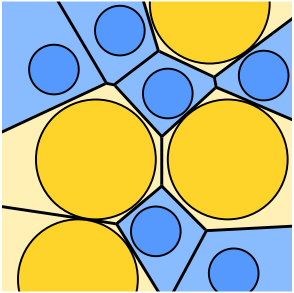
Blind bound: proof idea

Voronoi diagram: $dist_D(X) = |OX|$



$\longrightarrow$

Power diagram: $\Pi_D(X) = |OX|^2 - R^2$



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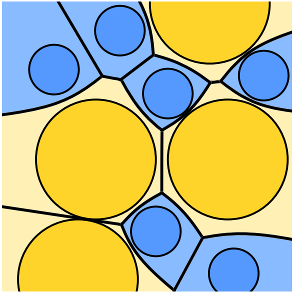

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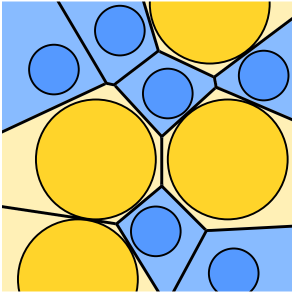
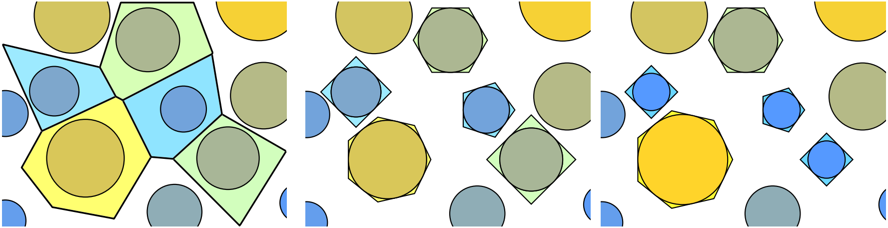
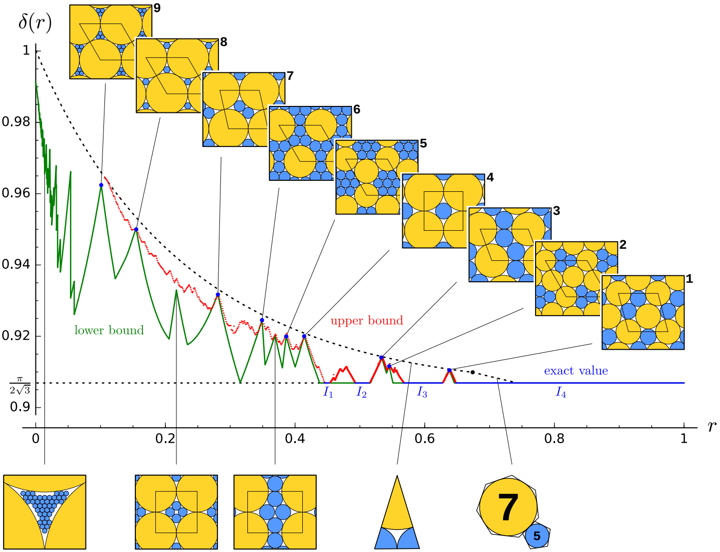




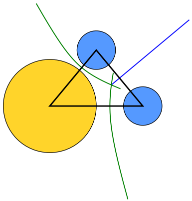
Fernique bounds



Density bounds: from discs to spheres

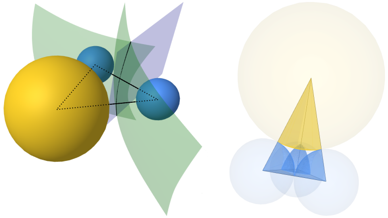
Upper density bound for -packings

FM-triangulation (triangles)

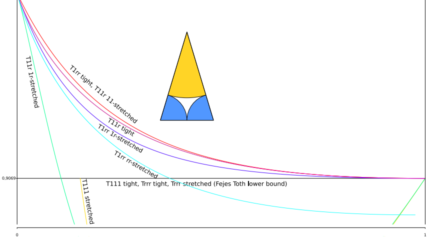


$\longrightarrow$

FM-simplicial partition (tetrahedra)

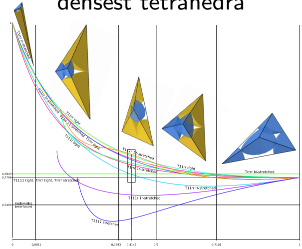


densest triangle



$\longrightarrow$

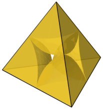
densest tetrahedra



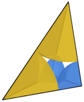
Upper density bound for -packings

Theorem, $r = \sqrt{2} - 1$ — in progress

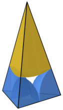
Each of the following tetrahedra is densest among the tetrahedra with the same spheres:



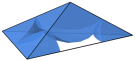
$\delta_{1111} \approx 0.7209$



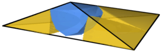
$\delta_{11rr} \approx 0.8105$



$\delta_{1rrr} \approx 0.8065$



$\delta_{rrrr} \approx 0.7847$



$\delta_{111r} \approx 0.8125$

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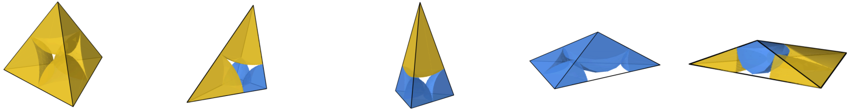

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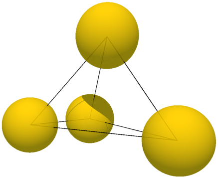
Proof:

- Reduce thde dimension of the set ($6 \to 5$): move spheres towards support sphere center



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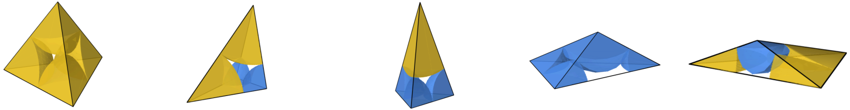

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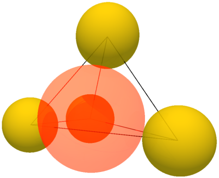

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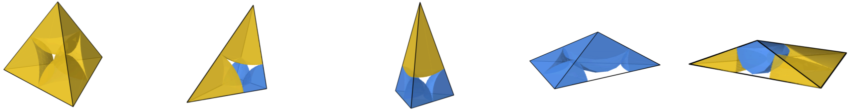

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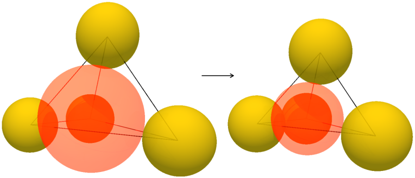

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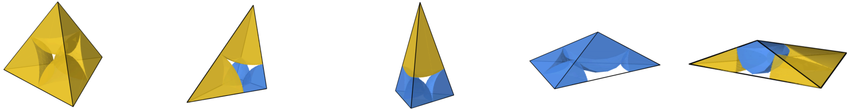

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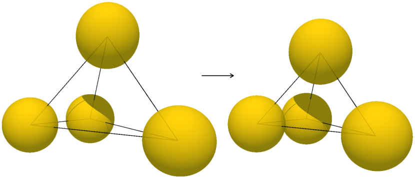

Upper density bound for ●●-packings

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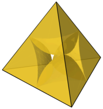
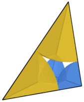
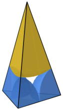
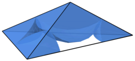
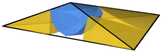
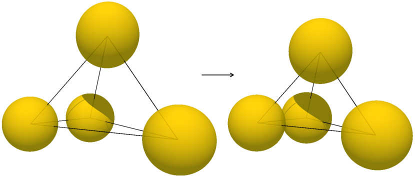






$\delta_{1111} \approx 0.7209$ $\delta_{11rr} \approx 0.8105$ $\delta_{1rrr} \approx 0.8065$ $\delta_{rrrr} \approx 0.7847$ $\delta_{111r} \approx 0.8125$

Proof:

- Reduce thde dimension of the set ($6 \to 5$): move spheres towards support sphere center



- Computer-assisted proof for tetrahedra with one contact: recursive subdivision + interval arithmetic
  $\approx$ 1000 lines of code

75 hours of CPU time

Why the computations are so slow

interval arithmetic $+$ huge formulas $\rightarrow$ loss of precision

Why the computations are so slow

interval arithmetic + huge formulas → loss of precision

Example: to compute the support sphere radius, we need to solve $A\,r^2 + B\,r + C = 0$

$$
\begin{aligned}
A = & -4a^2b^2d^2+4a^2c^2d^2+4b^2c^2d^2-4c^4d^2-4c^2d^4+4a^2b^2e^2-4b^4e^2-4a^2c^2e^2+4b^2c^2e^2+4b^2d^2e^2+4c^2d^2e^2-4b^2e^4-4a^4f^2+4a^2b^2f^2+4a^2c^2f^2-4b^2c^2f^2+4a^2d^2f^2+4c^2d^2f^2+4a^2e^2f^2+4b^2e^2f^2-4d^2e^2f^2\\
& -4a^2f^4+4d^4r_x^2-8d^2e^2r_x^2+4e^4r_x^2-8d^2f^2r_x^2-8e^2f^2r_x^2+4f^4r_x^2-8b^2d^2r_xr_y+8c^2d^2r_xr_y+8b^2e^2r_xr_y-8c^2e^2r_xr_y-16a^2f^2r_xr_y+8b^2f^2r_xr_y+8c^2f^2r_xr_y+8d^2f^2r_xr_y+8e^2f^2r_xr_y-8f^4r_xr_y+4b^4r_y^2-8b^2c^2r_y^2+4c^4r_y^2\\
& -8b^2f^2r_y^2-8c^2f^2r_y^2+4f^4r_y^2-8a^2d^2r_xr_z+8c^2d^2r_xr_z+8a^2e^2r_xr_z-16b^2e^2r_xr_z+8c^2e^2r_xr_z+8d^2e^2r_xr_z-8e^4r_xr_z+8a^2f^2r_xr_z-8c^2f^2r_xr_z+8e^2f^2r_xr_z-8a^2b^2r_yr_z+8a^2c^2r_yr_z+8b^2c^2r_yr_z-8c^4r_yr_z-16c^2d^2r_yr_z\\
& +8b^2e^2r_yr_z+8c^2e^2r_yr_z+8a^2f^2r_yr_z+8c^2f^2r_yr_z-8e^2f^2r_yr_z+4a^4r_z^2-8a^2c^2r_z^2+4c^4r_z^2-8a^2e^2r_z^2-8c^2e^2r_z^2+4e^4r_z^2+8a^2d^2r_xr_w+8b^2d^2r_xr_w-16c^2d^2r_xr_w-8d^4r_xr_w-8a^2e^2r_xr_w+8b^2e^2r_xr_w+8d^2e^2r_xr_w\\
& +8a^2f^2r_xr_w-8b^2f^2r_xr_w+8d^2f^2r_xr_w+8a^2b^2r_yr_w-8b^4r_yr_w-8a^2c^2r_yr_w+8b^2c^2r_yr_w+8b^2d^2r_yr_w+8c^2d^2r_yr_w-16b^2e^2r_yr_w+8a^2f^2r_yr_w+8b^2f^2r_yr_w-8d^2f^2r_yr_w-8a^4r_zr_w+8a^2b^2r_zr_w+8a^2c^2r_zr_w-8b^2c^2r_zr_w\\
& +8a^2d^2r_zr_w+8c^2d^2r_zr_w+8a^2e^2r_zr_w+8b^2e^2r_zr_w-8d^2e^2r_zr_w-16a^2f^2r_zr_w+4a^4r_w^2-8a^2b^2r_w^2+4b^4r_w^2-8a^2d^2r_w^2-8b^2d^2r_w^2+4d^4r_w^2
\end{aligned}
$$

$$
\begin{aligned}
B = & -4c^2d^4r_x+4b^2d^2e^2r_x+4c^2d^2e^2r_x-4b^2e^4r_x+4a^2d^2f^2r_x+4c^2d^2f^2r_x+4a^2e^2f^2r_x+4b^2e^2f^2r_x-8d^2e^2f^2r_x-4a^2f^4r_x+4d^4r_x^3-8d^2e^2r_x^3+4e^4r_x^3-8d^2f^2r_x^3-8e^2f^2r_x^3+4f^4r_x^3+4b^2c^2d^2r_y-4c^4d^2r_y-4b^4e^2r_y\\
& +4b^2c^2e^2r_y+4a^2b^2f^2r_y+4a^2c^2f^2r_y-8b^2c^2f^2r_y+4c^2d^2f^2r_y+4b^2e^2f^2r_y-4a^2f^4r_y-4b^2d^2r_x^2r_y+4c^2d^2r_x^2r_y+4b^2e^2r_x^2r_y-4c^2e^2r_x^2r_y-8a^2f^2r_x^2r_y+4b^2f^2r_x^2r_y+4c^2f^2r_x^2r_y+4d^2f^2r_x^2r_y+4e^2f^2r_x^2r_y-4f^4r_x^2r_y\\
& -4b^2d^2r_xr_y^2+4c^2d^2r_xr_y^2+4b^2e^2r_xr_y^2-4c^2e^2r_xr_y^2-8a^2f^2r_xr_y^2+4b^2f^2r_xr_y^2+4c^2f^2r_xr_y^2+4d^2f^2r_xr_y^2+4e^2f^2r_xr_y^2-4f^4r_xr_y^2+4b^4r_y^3-8b^2c^2r_y^3+4c^4r_y^3-8b^2f^2r_y^3-8c^2f^2r_y^3+4f^4r_y^3+4a^2c^2d^2r_z-4c^4d^2r_z+4a^2b^2e^2r_z\\
& -8a^2c^2e^2r_z+4b^2c^2e^2r_z+4c^2d^2e^2r_z-4b^2e^4r_z-4a^4f^2r_z+4a^2c^2f^2r_z+4a^2e^2f^2r_z-4a^2d^2r_x^2r_z+4c^2d^2r_x^2r_z+4a^2e^2r_x^2r_z-8b^2e^2r_x^2r_z+4c^2e^2r_x^2r_z+4d^2e^2r_x^2r_z-4e^4r_x^2r_z+4a^2f^2r_x^2r_z-4c^2f^2r_x^2r_z+4e^2f^2r_x^2r_z\\
& -4a^2b^2r_y^2r_z+4a^2c^2r_y^2r_z+4b^2c^2r_y^2r_z-4c^4r_y^2r_z-8c^2d^2r_y^2r_z+4b^2e^2r_y^2r_z+4c^2e^2r_y^2r_z+4a^2f^2r_y^2r_z+4c^2f^2r_y^2r_z-4e^2f^2r_y^2r_z-4a^2d^2r_xr_z^2+4c^2d^2r_xr_z^2+4a^2e^2r_xr_z^2-8b^2e^2r_xr_z^2+4c^2e^2r_xr_z^2+4d^2e^2r_xr_z^2-4e^4r_xr_z^2\\
& +4a^2f^2r_xr_z^2-4c^2f^2r_xr_z^2+4e^2f^2r_xr_z^2-4a^2b^2r_yr_z^2+4a^2c^2r_yr_z^2+4b^2c^2r_yr_z^2-4c^4r_yr_z^2-8c^2d^2r_yr_z^2+4b^2e^2r_yr_z^2+4c^2e^2r_yr_z^2+4a^2f^2r_yr_z^2+4c^2f^2r_yr_z^2-4e^2f^2r_yr_z^2+4a^4r_z^3-8a^2c^2r_z^3+4c^4r_z^3-8a^2e^2r_z^3-8c^2e^2r_z^3\\
& +4e^4r_z^3-8a^2b^2d^2r_w+4a^2c^2d^2r_w+4b^2c^2d^2r_w-4c^2d^4r_w+4a^2b^2e^2r_w-4b^4e^2r_w+4b^2d^2e^2r_w-4a^4f^2r_w+4a^2b^2f^2r_w+4a^2d^2f^2r_w+4a^2d^2r_x^2r_w+4b^2d^2r_x^2r_w-8c^2d^2r_x^2r_w-4d^4r_x^2r_w-4a^2e^2r_x^2r_w+4b^2e^2r_x^2r_w\\
& +4d^2e^2r_x^2r_w+4a^2f^2r_x^2r_w-4b^2f^2r_x^2r_w+4d^2f^2r_x^2r_w+4a^2b^2r_y^2r_w-4b^4r_y^2r_w-4a^2c^2r_y^2r_w+4b^2c^2r_y^2r_w+4b^2d^2r_y^2r_w+4c^2d^2r_y^2r_w-8b^2e^2r_y^2r_w+4a^2f^2r_y^2r_w+4b^2f^2r_y^2r_w-4d^2f^2r_y^2r_w-4a^4r_z^2r_w+4a^2b^2r_z^2r_w+4a^2c^2r_z^2r_w\\
& -4b^2c^2r_z^2r_w+4a^2d^2r_z^2r_w+4c^2d^2r_z^2r_w+4a^2e^2r_z^2r_w+4b^2e^2r_z^2r_w-4d^2e^2r_z^2r_w-8a^2f^2r_z^2r_w+4a^2d^2r_xr_w^2+4b^2d^2r_xr_w^2-8c^2d^2r_xr_w^2-4d^4r_xr_w^2-4a^2e^2r_xr_w^2+4b^2e^2r_xr_w^2+4d^2e^2r_xr_w^2+4a^2f^2r_xr_w^2-4b^2f^2r_xr_w^2\\
& +4d^2f^2r_xr_w^2+4a^2b^2r_yr_w^2-4b^4r_yr_w^2-4a^2c^2r_yr_w^2+4b^2c^2r_yr_w^2+4b^2d^2r_yr_w^2+4c^2d^2r_yr_w^2-8b^2e^2r_yr_w^2+4a^2f^2r_yr_w^2+4b^2f^2r_yr_w^2-4d^2f^2r_yr_w^2-4a^4r_zr_w^2+4a^2b^2r_zr_w^2+4a^2c^2r_zr_w^2\\
& -4b^2c^2r_zr_w^2+4a^2d^2r_zr_w^2+4c^2d^2r_zr_w^2+4a^2e^2r_zr_w^2+4b^2e^2r_zr_w^2-4d^2e^2r_zr_w^2-8a^2f^2r_zr_w^2+4a^4r_w^3-8a^2b^2r_w^3+4b^4r_w^3-8a^2d^2r_w^3-8b^2d^2r_w^3+4d^4r_w^3
\end{aligned}
$$

$$
\begin{aligned}
C = & c^4d^4-2b^2c^2d^2e^2+b^4e^4-2a^2c^2d^2f^2-2a^2b^2e^2f^2+a^4f^4-2c^2d^4r_x^2+2b^2d^2e^2r_x^2+2c^2d^2e^2r_x^2-2b^2e^4r_x^2+2a^2d^2f^2r_x^2+2c^2d^2f^2r_x^2+2a^2e^2f^2r_x^2+2b^2e^2f^2r_x^2-4d^2e^2f^2r_x^2-2a^2f^4r_x^2+d^4r_x^4-2d^2e^2r_x^4+e^4r_x^4\\
& -2d^2f^2r_x^4-2e^2f^2r_x^4+f^4r_x^4+2b^2c^2d^2r_y^2-2c^4d^2r_y^2-2b^4e^2r_y^2+2b^2c^2e^2r_y^2+2a^2b^2f^2r_y^2+2a^2c^2f^2r_y^2-4b^2c^2f^2r_y^2+2c^2d^2f^2r_y^2+2b^2e^2f^2r_y^2-2a^2f^4r_y^2-2b^2d^2r_x^2r_y^2+2c^2d^2r_x^2r_y^2+2b^2e^2r_x^2r_y^2-2c^2e^2r_x^2r_y^2-4a^2f^2r_x^2r_y^2\\
& +2b^2f^2r_x^2r_y^2+2c^2f^2r_x^2r_y^2+2d^2f^2r_x^2r_y^2+2e^2f^2r_x^2r_y^2-2f^4r_x^2r_y^2+b^4r_y^4-2b^2c^2r_y^4+c^4r_y^4-2b^2f^2r_y^4-2c^2f^2r_y^4+f^4r_y^4+2a^2c^2d^2r_z^2-2c^4d^2r_z^2+2a^2b^2e^2r_z^2-4a^2c^2e^2r_z^2+2b^2c^2e^2r_z^2+2c^2d^2e^2r_z^2-2b^2e^4r_z^2\\
& -2a^4f^2r_z^2+2a^2c^2f^2r_z^2+2a^2e^2f^2r_z^2-2a^2d^2r_x^2r_z^2+2c^2d^2r_x^2r_z^2+2a^2e^2r_x^2r_z^2-4b^2e^2r_x^2r_z^2+2c^2e^2r_x^2r_z^2+2d^2e^2r_x^2r_z^2-2e^4r_x^2r_z^2+2a^2f^2r_x^2r_z^2-2c^2f^2r_x^2r_z^2+2e^2f^2r_x^2r_z^2-2a^2b^2r_y^2r_z^2+2a^2c^2r_y^2r_z^2+2b^2c^2r_y^2r_z^2\\
& -2c^4r_y^2r_z^2-4c^2d^2r_y^2r_z^2+2b^2e^2r_y^2r_z^2+2c^2e^2r_y^2r_z^2+2a^2f^2r_y^2r_z^2+2c^2f^2r_y^2r_z^2-2e^2f^2r_y^2r_z^2+a^4r_z^4-2a^2c^2r_z^4+c^4r_z^4-2a^2e^2r_z^4-2c^2e^2r_z^4+e^4r_z^4-4a^2b^2d^2r_w^2+2a^2c^2d^2r_w^2+2b^2c^2d^2r_w^2-2c^2d^4r_w^2+2a^2b^2e^2r_w^2\\
& -2b^4e^2r_w^2+2b^2d^2e^2r_w^2-2a^4f^2r_w^2+2a^2b^2f^2r_w^2+2a^2d^2f^2r_w^2+2a^2d^2r_x^2r_w^2+2b^2d^2r_x^2r_w^2-4c^2d^2r_x^2r_w^2-2d^4r_x^2r_w^2-2a^2e^2r_x^2r_w^2+2b^2e^2r_x^2r_w^2+2d^2e^2r_x^2r_w^2+2a^2f^2r_x^2r_w^2-2b^2f^2r_x^2r_w^2+2d^2f^2r_x^2r_w^2+2a^2b^2r_y^2r_w^2\\
& -2b^4r_y^2r_w^2-2a^2c^2r_y^2r_w^2+2b^2c^2r_y^2r_w^2+2b^2d^2r_y^2r_w^2+2c^2d^2r_y^2r_w^2-4b^2e^2r_y^2r_w^2+2a^2f^2r_y^2r_w^2+2b^2f^2r_y^2r_w^2-2d^2f^2r_y^2r_w^2-2a^4r_z^2r_w^2+2a^2b^2r_z^2r_w^2+2a^2c^2r_z^2r_w^2-2b^2c^2r_z^2r_w^2+2a^2d^2r_z^2r_w^2+2c^2d^2r_z^2r_w^2\\
& +2a^2e^2r_z^2r_w^2+2b^2e^2r_z^2r_w^2-2d^2e^2r_z^2r_w^2-4a^2f^2r_z^2r_w^2+a^4r_w^4-2a^2b^2r_w^4+b^4r_w^4-2a^2d^2r_w^4-2b^2d^2r_w^4+d^4r_w^4
\end{aligned}
$$

Why the computations are so slow

interval arithmetic + huge formulas → loss of precision

Example: to compute the support sphere radius, we need to solve $A\,r^2 + B\,r + C = 0$

Thanks to dimension reduction:
compute with fixed radii and edge lengths, then "simplify"

$r_x = r_y = r_z = r_w = 1,\quad a = 2$:

$$A = 4\left(b\left((c-4)(d-f)-(b-c-d+e-f-4)\,e\right)-\left((c+d-e-f-4)\,d+4\,(e-f)\right)c-f\left(d\,(e-4)-4\,e+4\,f+16\right)\right)$$

$$B = 8\left(b\left((c-4)(d-f)-(b-c-d+e-f-4)\,e\right)-\left((c+d-e-f-4)\,d+4\,(e-f)\right)c-f\left(d\,(e-4)-4\,e+4\,f+16\right)\right)$$

$$C = b\left(4\,(c-4)(d-f)-e\left(2\,c\,(d-2)-b\,(e-4)-4\,(d+e+f-4)\right)\right)+(d-4)\left(c\left(d\,(c-4)+4\,(e-f)\right)-4\,f\,(e-4)\right)$$

Steps of the proof

δ^* denotes the maximal density

1. partition space into "small" cells — FM-triangulation
2. find a suitable function to represent the density — emptiness E

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4. verify that the redistributed density $\leq \delta^*$ — (•) choice of m in $\dot{U}$
   check all possible local configurations — run through all coronas
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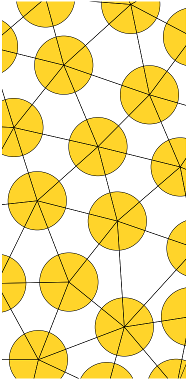
---

6. verify that the sum of vertex densities in a cell $\geq$ its density — interval arithmetic
   check all possible cells

Space partition

2D,

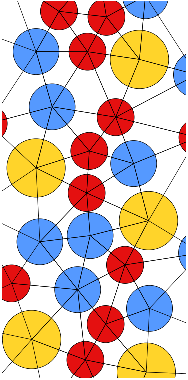
Delaunay triangulation



2D,

FM-triangulation $\mathcal{T}$

(weighted Delaunay triangulation)



3D,

$\underbrace{\text{Voronoi cells} + \text{Delaunay simplices}}$

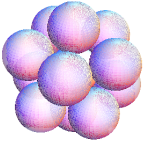
HF-partition $\mathcal{P}$

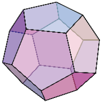
consists of several types of simplices and modified Voronoi cells

(Sphere packings II. A formulation of the Kepler Conjecture)

only Voronoi cells or
only Delaunay simplices
$\Downarrow$
local configurations denser than δ^*

regular dodecahedron for Voronoi cells
(as in dodecahedral conjecture)



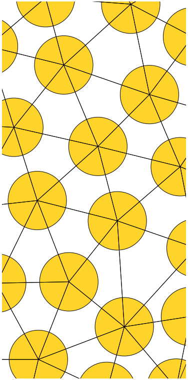


Hales and McLaughlin 1998

Space partition

2D,

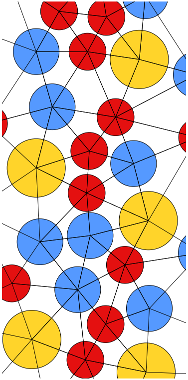
Delaunay triangulation



2D,

FM-triangulation $\mathcal{T}$

(weighted Delaunay triangulation)



3D,

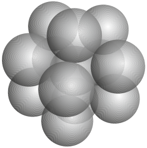
$$\underbrace{\text{Voronoi cells} + \text{Delaunay simplices}}_{\text{HF-partition } \mathcal{P}}$$

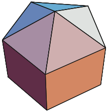
consists of several types of simplices and modified Voronoi cells

(Sphere packings II. A formulation of the Kepler Conjecture)

only Voronoi cells or
only Delaunay simplices
$\Downarrow$
local configurations denser than δ^*

pentagonal prism for Delaunay simplices





Ferguson 2006

Suitable function to measure the density

2D	2D	3D
density δ	emptiness	compression
	$E(\Delta) := \delta^* \times area(\Delta) - area(\Delta \cap P)$	$\Gamma(R) := vol(R \cap P) - \delta_{\text{oct}} \times vol(R)$
		$\delta_{\text{oct}} := \delta(\text{tight regular octahedron}) < \delta^*$
		$\delta^* = \frac{\delta_{\text{tet}}}{3} + 2\frac{\delta_{\text{oct}}}{3}$

Suitable function to measure the density

2D,

density
δ

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emptiness

$E(\Delta) := \delta^* \times area(\Delta) - area(\Delta \cap P)$

3D,

compression

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$\delta^* = \frac{\delta_{\text{tet}}}{3} + 2\frac{\delta_{\text{oct}}}{3}$

An additive function reflecting the density:

$\Gamma(R) < 0$ iff the density of R is less than δ_{oct}
$\Gamma(R) > 0$ iff the density of R is greater than δ_{oct}

$\delta \leq \delta^* \Leftarrow \Gamma$ is "low enough" on each small region

Redistribution of emptiness and compression

2D,

vertex potential

for $\Delta \in \mathcal{T}, v \in \Delta$,
$\dot{U}_\Delta^v = V_{xyz} + m \times |\hat{v} - \hat{v}_{xyz}|$

where x, y, z are disc radii of Δ
$\hat{v}$ is the angle of Δ in v
and $V_{xyz}, m, \hat{v}_{xyz}$ are constants

3D,

score

for $R \in \mathcal{P}, v \in R$,
$\sigma(R, v)$ depends on the type of R

if R is a Voronoi cell of v $(R = Vor(v))$,
then $\sigma(R, v) = 4\Gamma(R)$
and $\sigma(R, w) = 0$ for $w \neq v$

if R is a simplex, σ varies in function of its properties and depends on Γ

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$$U(\Delta) = U^A_\Delta + U^B_\Delta + U^C_\Delta \leq E(\Delta)$$

i.e. U is easy to manipulate and is at most E (lower approximation)

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$$\sum_{v \in R} \sigma(R, v) = 4\Gamma(R)$$

i.e. score of a region always equals to $4\times$compression

(Sphere packings II. A formulation of the Kepler Conjecture)

Verify that redistribution $\leq \delta^*$ around each vertex

2D,

for each $v \in \mathcal{T}$,

$$\sum_{\Delta \in C_v} \dot{U}^v_\Delta \geq 0 \qquad (\bullet)$$

configuration around a vertex – corona C

combinatorial representation of C – sequence of disc radii $S(C)$

choose m to satisfy

$$\sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C}} V_{xyz} + m \times \Big|2\pi - \sum_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in C}} \widehat{xyz}\Big| \geq 0$$

for all coronas C

FM-triangulation $\Rightarrow$ bounded $|S(C)|$
$\Rightarrow$ finite number of inequalities on m
$\Rightarrow$ computer search

3D,

for each $v \in \mathcal{P}$, $\sum_{R \in D_v} \sigma(R, v) \leq 8pt$

configuration around a vertex – decomposition star D

combinatorial representation of D – graph $G(D)$

conditions on geometry of D "easily" implying $\sigma(D) \leq 8pt$ (interval arithmetics)

(Sphere packings IV. Detailed bounds)

tame graphs – 25 000 graphs of the remaining D have restricted geometry $\Rightarrow \max \sigma(D) \leq 8pt$ (linear programming)

(Sphere packings VI. Tame graphs and linear programs)

except **pentagonal prism graph**,

(Sphere packing V. Pentahedral prisms)

FCC graph and **HCP graph**

(Sphere packings III. Extremal cases)

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Extremal cases

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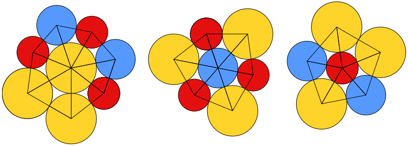

for the coronas of $\mathcal{T}^*$, $\sum\limits_{\substack{x,y,z \\ \text{disc radii of} \\ \Delta \in y\text{-corona}}} V_{xyz} := 0$

for tight triangles, $U(\Delta_{xyz}) := E(\Delta_{xyz})$

ϵ-triangles – triangles close to tight
$\Rightarrow$ potential close to emptiness

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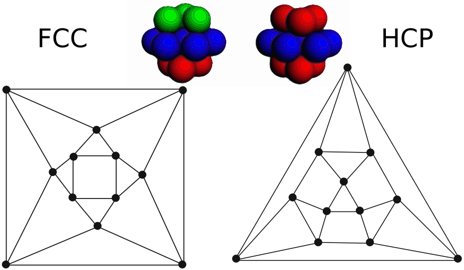
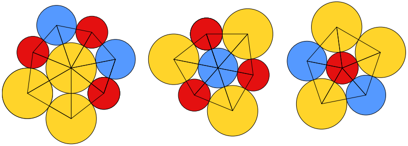


FCC and HCP decomposition stars have maximal score
$\Rightarrow$ close configurations have high score

Extremal cases

2D,



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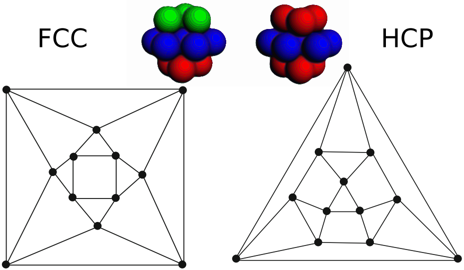
ϵ-triangles – triangles close to tight
$\Rightarrow$ potential close to emptiness

derivatives on side lengths x_i:

$$\min_{T_\epsilon} \frac{\partial E}{\partial x_i}\Delta x_i \geq \max_{T_\epsilon} \frac{\partial U}{\partial x_i}\Delta x_i$$

(interval arithmetic)

3D,

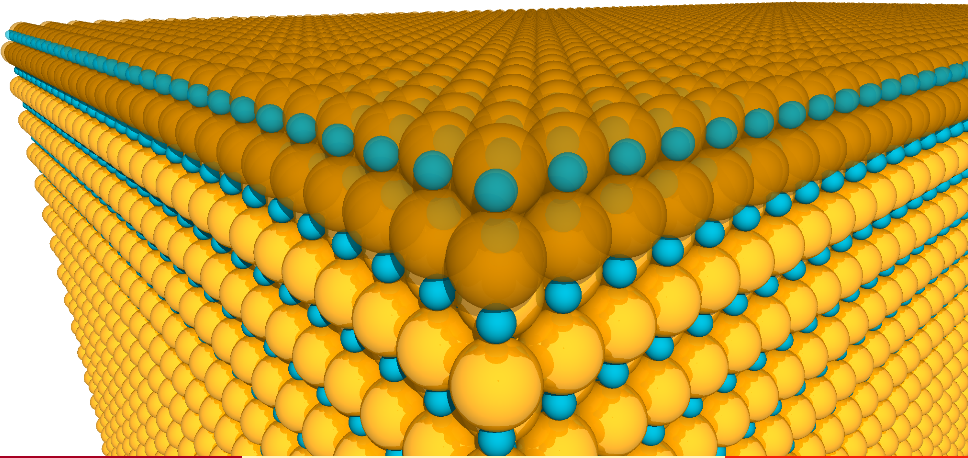


FCC and HCP decomposition stars have maximal score
$\Rightarrow$ close configurations have high score

derivatives to prove that FCC and HCP are local maxima

(Sphere packings III. Extremal cases)

Do the same for ?



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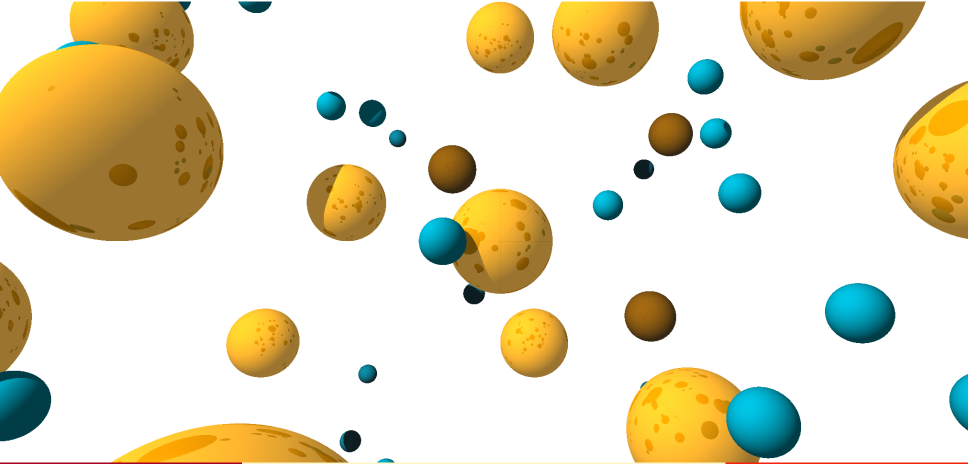
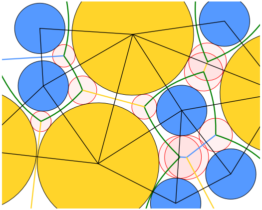


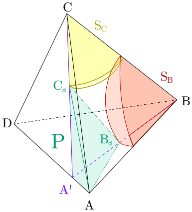
1. Sphere packings in 3D
2. Density bounds: from discs to spheres
3. Optimality proofs for
4. Conclusion

Techniques

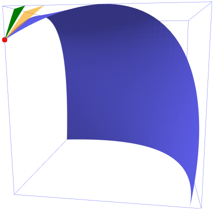
s **Geometry:**



properties of triangulations

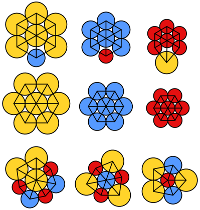


... and "tetrahedrizations"



differential geometry

Computer assistance:



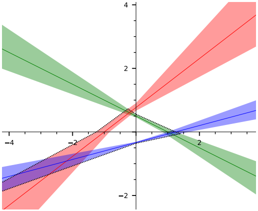
case enumeration
Python, C++

$$\mathcal{A}_{1111} = 4\,\left(d^2-e^2\right)^2 + 4\,f^4 + \left(\left(d^2-8\right)e^2 - 8\,d^2\right)f^2$$
$$\mathcal{B}_{1111} = 8\,\left(d^2-e^2\right)^2 + 8\,f^4 + 2\left(\left(d^2-8\right)e^2 - 8\,d^2\right)f^2$$
$$\mathcal{C}_{1111} = d^2e^2f^2$$

$$\delta_{1111} = \frac{8\left(\begin{array}{r} \arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{(d+2)e^2+2\,d^2+(d^2+4\,d)e-2\,f^2}\right) \\ +\arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{(d+2)f^2+2\,d^2-2\,e^2+(d^2+4\,d)f}\right) \\ +\arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{(e+2)f^2-2\,d^2+2\,e^2+(e^2+4\,e)f}\right) \\ -\arctan\left(\frac{\sqrt{-4\,(d^2-e^2)^2-4\,f^4-((d^2-8)e^2-8\,d^2)f^2}}{2\,(d^2+e^2+f^2-32)}\right) \end{array}\right)}{\sqrt{-4\,\left(d^2-e^2\right)^2-4\,f^4-\left(\left(d^2-8\right)e^2-8\,d^2\right)f^2}}.$$

symbolic calculus
SageMath

$\Longleftrightarrow$

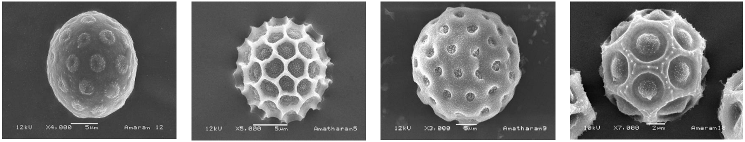


interval arithmetic
MPFI (RIF SageMath)
Boost (C++)

Thank you for your attention!

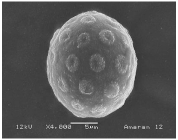
5. Other “spherical” questions: from pollen grains to kissing problem

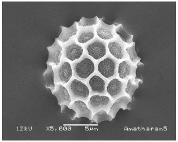
Tammes 1930: configuration of pores on a pollen grain **spherical codes**

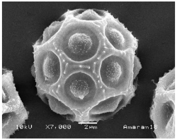


maximize the number of spherical caps of a given radius on a sphere

Tammes 1930: configuration of pores on a pollen grain **spherical codes**







maximize the number of spherical caps of a given radius on a sphere

place n points on a sphere to maximize the distance between two nearest points

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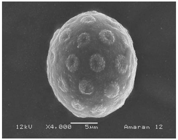

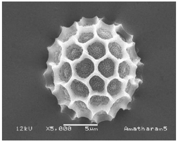

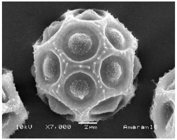

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find the smallest possible radius of a central sphere tangent to n unit spheres

solved for $n = 3, \ldots, 14$, and 24 (1943 – 2015)

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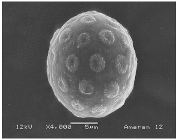

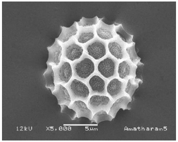

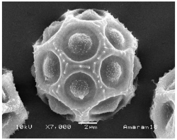

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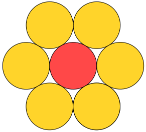
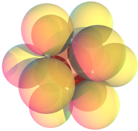
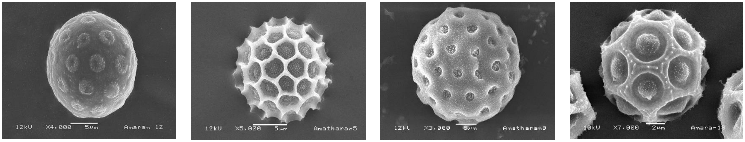
solved for $d = 2 : 6$, $3 :$

Other "spherical" questions: from pollen grains to kissing problem

Tammes 1930: configuration of pores on a pollen grain **spherical codes**



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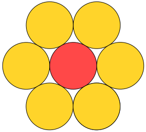
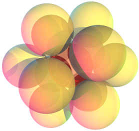
solved for $d = 2 : 6$, $3 : 12$ (1953), $4 : 24$ (2003), $8 : 240$, $24 : 196560$ (1979)





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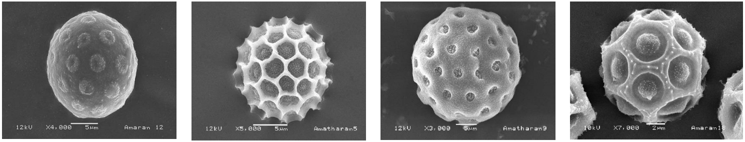

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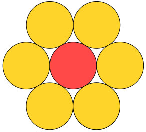

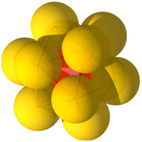

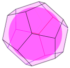

dodecahedral conjecture
smallest Voronoi cell in sphere packing
(proved in 2010)