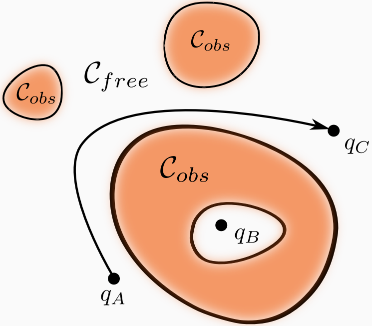
Algorithms for connectivity queries on unbounded real algebraic sets

14$^{\text{th}}$ October 2025

JGA '25



Rémi Prébet
Joint works with M. Safey El Din, É. Schost
J. Capco, P. Wenger

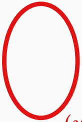
Slides:
rprebet.github.io/#talks

Algorithms for polynomial systems with real variables

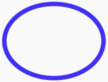
Semi-algebraic sets

Real solutions of systems of **polynomial equations** and **inequalities**

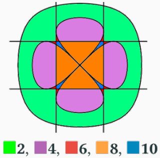
$$\begin{cases} 4y + x^3 - 4x^2 - 2x - 8 = 0 \\ -2 \leq x \leq 0 \end{cases}$$



$$\frac{x^2}{4} + y^2 - 1 = 0$$



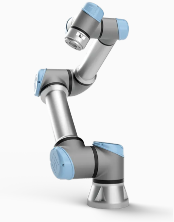
$$(x-1)^2 + \frac{(y-1)^2}{9} - 1 = 0$$



Physics

```
@real: i, s, n;
@pre: n ≥ 1;
i = 0; s = 0;
@invariant: [[{i,s},2]] ≥ 0 ∧ [[{i,s},2]] ≥ 0
while( [[{i,n},1]] )
{
        i = i+1;
        s = [[{i,s},1]];
}
@post: (n−1)·n/2 ≤ s ≤ n·(n+1)/2;
```

Program verification

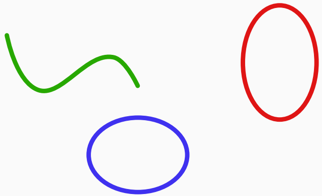


Robotics

Algorithms for polynomial systems with real variables

Semi-algebraic sets

Real solutions of systems of **polynomial equations** and **inequalities**



Fundamental algorithmic problems

Project: what is the possible set of values?

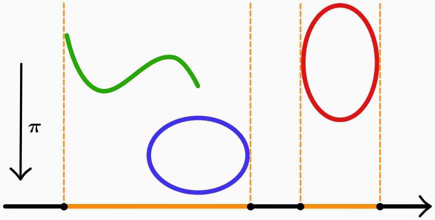
Sample: are there any solutions?

Connect: are two points connected?

Algorithms for polynomial systems with real variables

Semi-algebraic sets

Real solutions of systems of **polynomial equations** and **inequalities**

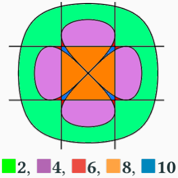


Fundamental algorithmic problems

Project: what is the possible set of values?

Sample: are there any solutions?

Connect: are two points connected?

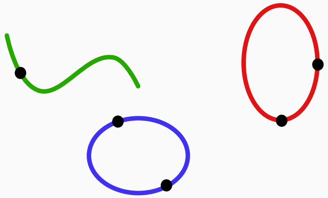


Physics

Algorithms for polynomial systems with real variables

Semi-algebraic sets

Real solutions of systems of **polynomial equations** and **inequalities**



Fundamental algorithmic problems

Project: what is the possible set of values?

Sample: are there any solutions?

Connect: are two points connected?

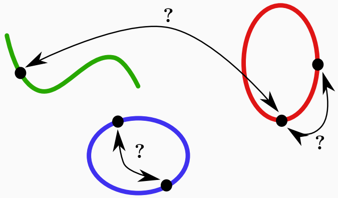
```
@real: i, s, n;
@pre: n ≥ 1;
i = 0; s = 0;
@invariant: [{i,s},2] ≥ 0  ∧  [{i,s},2] ≥ 0
while( [{i,n},1] )
{
        i = i + 1;
        s = [{i,s},1];
}
@post: (n − 1) · n/2 ≤ s ≤ n · (n + 1)/2;
```

Program verification

Algorithms for polynomial systems with real variables

Semi-algebraic sets

Real solutions of systems of **polynomial equations** and **inequalities**

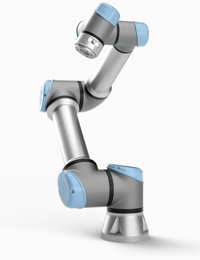


Fundamental algorithmic problems

Project: what is the possible set of values?

Sample: are there any solutions?

Connect: are two points connected?

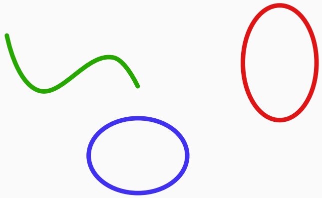


Robotics

Algorithms for polynomial systems with real variables

Semi-algebraic sets

Real solutions of systems of **polynomial equations** and **inequalities**

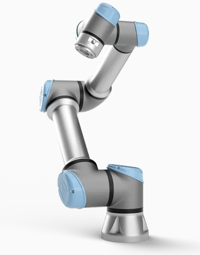


Fundamental algorithmic problems

Project: what is the possible set of values?

Sample: are there any solutions?

Connect: are two points connected?



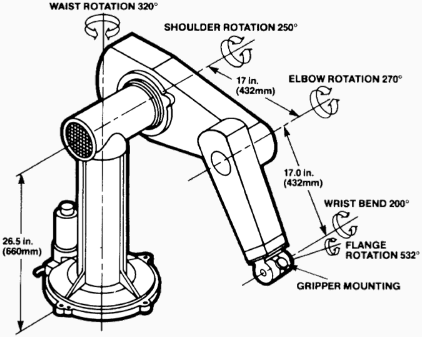
Robotics

A challenging application in robotics

$\mathrm{Jac}_{v_2,\ldots,v_5}(\mathcal{K})$ for a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix}
(v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\
0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\
0 & 1 & 0 & 0 & 0 & 2a_3 \\
0 & 0 & 1 & 0 & 0 & 0 \\
v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\
(v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1)
\end{bmatrix}$$

where $A(\boldsymbol{v}) = (v_3^2-1)(v_2^2-1)-4v_2v_3$



A PUMA 560 [Unimation, 1984]

Robotic problem

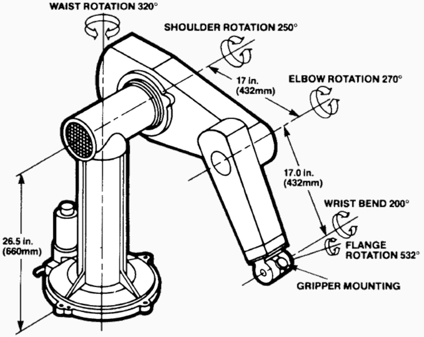
Count the number of aspects of this robot.

A challenging application in robotics

$\mathrm{Jac}_{v_2,\dots,v_5}(\mathcal{K})$ for a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix}
(v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1)\\
0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3\\
0 & 1 & 0 & 0 & 0 & 2a_3\\
0 & 0 & 1 & 0 & 0 & 0\\
v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0\\
(v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1)
\end{bmatrix}$$

where $A(\boldsymbol{v}) = (v_3^2-1)(v_2^2-1)-4v_2v_3$



A PUMA 560 [Unimation, 1984]

Robotic problem

Count the number of aspects of this robot.

$\Updownarrow$

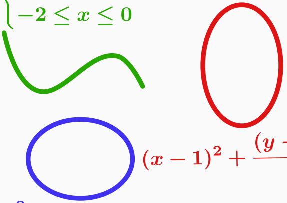
Semi-algebraic problem

Compute the number of connected components of $S = \{\boldsymbol{v} \in \mathbb{R}^4 \mid \det(M(\boldsymbol{v})) \neq 0\}$

Semi-algebraic set S

Defined by s polynomial (equations+inequalities) with n variables of deg $\leq D$

$$\begin{cases} 4y + x^3 - 4x^2 - 2x - 8 = 0 \\ -2 \leq x \leq 0 \end{cases}$$



$$(x-1)^2 + \frac{(y-1)^2}{9} - 1 = 0$$

$$\frac{x^2}{4} + y^2 - 1 = 0$$

General-purpose algorithm [Collins; '75]

Complexity: $(sD)^{2^{O(n)}}$

Fundamental algorithmic problem

Project S onto t coordinates $\rightsquigarrow s^{n+1} D^{O(nt)}$ [Basu-Pollack-Roy, '96] **Optimal**

Sample points of S $\rightsquigarrow s^{n} D^{O(n)}$ [Basu-Pollack-Roy, '98] **Optimal**

Connect two points in S $\rightsquigarrow s^{n+1} D^{O(n^2)}$ [Basu-Pollack-Roy, '00] **Not optimal**

Computing connectivity properties: Roadmaps

[Canny, 1988] Compute $\mathscr{R} \subset S$ one-dimensional, sharing its connectivity

Roadmap of $(S, \mathcal{P})$

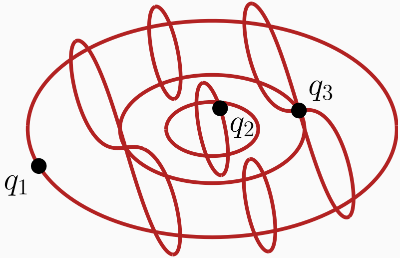
A *semi-algebraic curve* $\mathscr{R} \subset S$, containing query points $(q_1, \ldots, q_N)$ s.t. for all connected components C of S: $C \cap \mathscr{R}$ is *non-empty* and *connected*

Proposition

q_i and q_j are path-connected in $S \iff$ they are in $\mathscr{R}$

Problem reduction

Arbitrary dimension $\underset{\text{ROADMAP}}{\Longrightarrow}$ Dimension 1



Computing connectivity properties: Roadmaps

[Canny, 1988] Compute $\mathscr{R} \subset S$ one-dimensional, sharing its connectivity

Roadmap of $(S, \mathcal{P})$

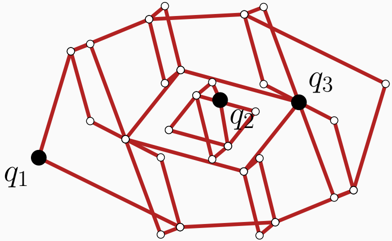
A *semi-algebraic curve* $\mathscr{R} \subset S$, containing query points $(q_1, \ldots, q_N)$ s.t. for all connected components C of S: $C \cap \mathscr{R}$ is *non-empty* and *connected*

Proposition

q_i and q_j are path-connected in $S \iff$ they are in $\mathscr{R} \iff$ they are in $\mathscr{G}$

Problem reduction

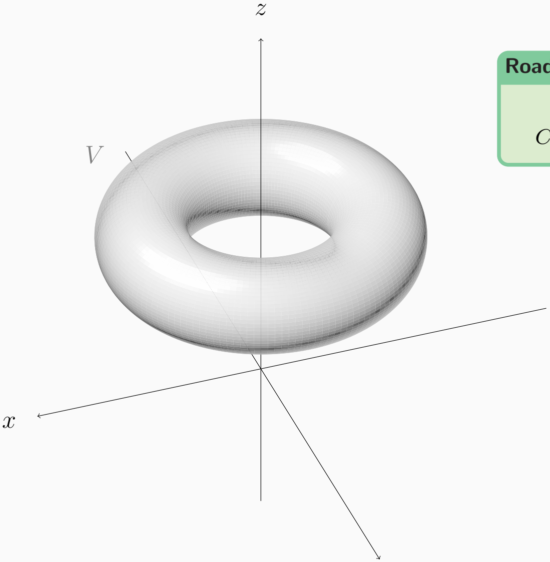
Arbitrary dimension $\underset{\text{ROADMAP}}{\Longrightarrow}$ Dimension 1 $\underset{\text{Topology}}{\Longrightarrow}$ Finite graph $\mathscr{G}$



Roadmap algorithms for unbounded algebraic sets

joint work with M. Safey El Din and É. Schost

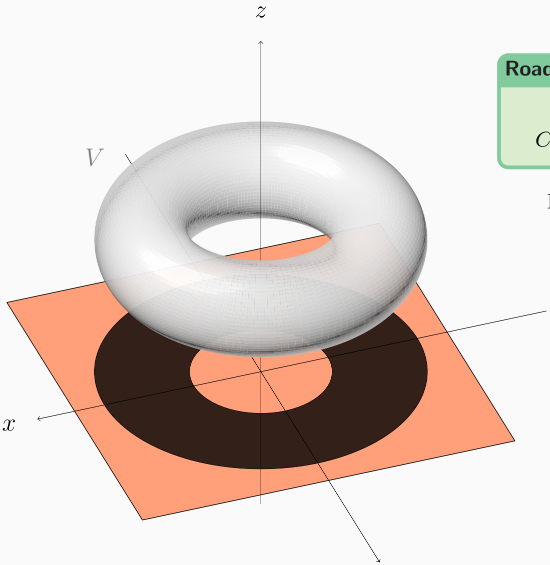
Canny's strategy



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

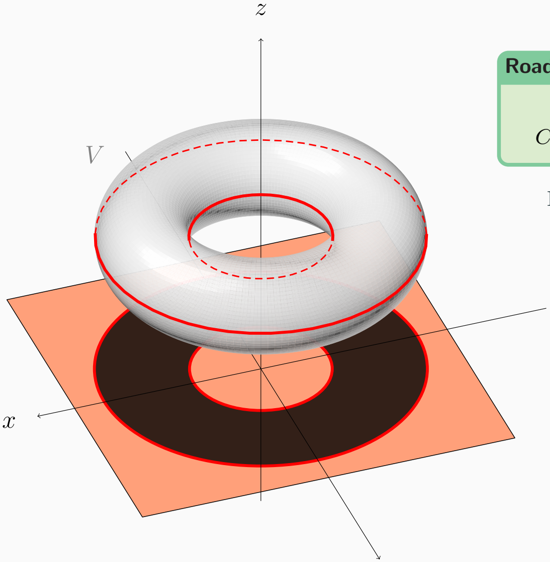
Canny's strategy



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Projection through:
$\pi_2 \colon (x_1, \ldots, x_n) \mapsto (x_1, x_2)$



Roadmap property

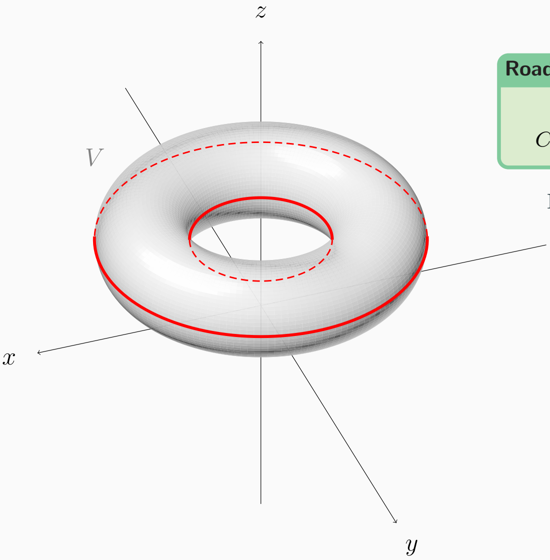
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Projection through:
$\pi_2 \colon (x_1, \ldots, x_n) \mapsto (x_1, x_2)$

$W(\pi_2, V)$ critical locus of π_2.

Intersects all the
connected components of V

Canny's strategy



Roadmap property

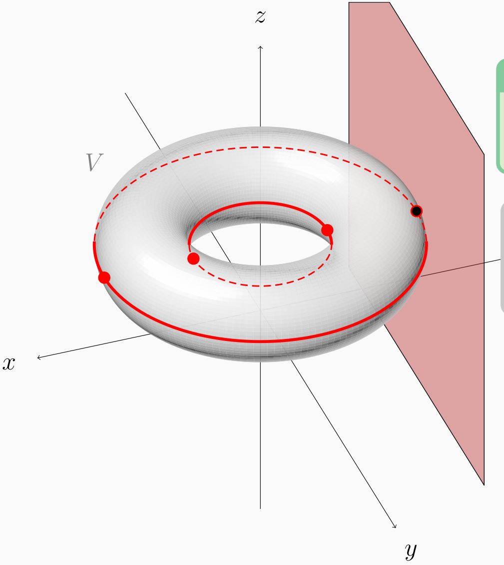
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Projection through:
$\pi_2 \colon (x_1, \ldots, x_n) \mapsto (x_1, x_2)$

$W(\pi_2, V)$ critical locus of π_2.

Intersects all the
connected components of V

Canny's strategy



Roadmap property

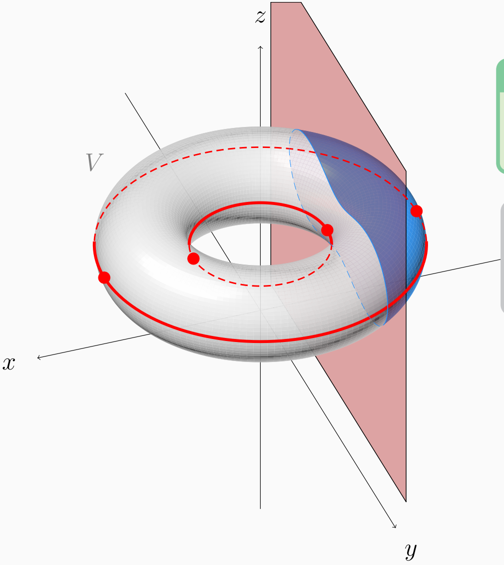
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

Canny's strategy



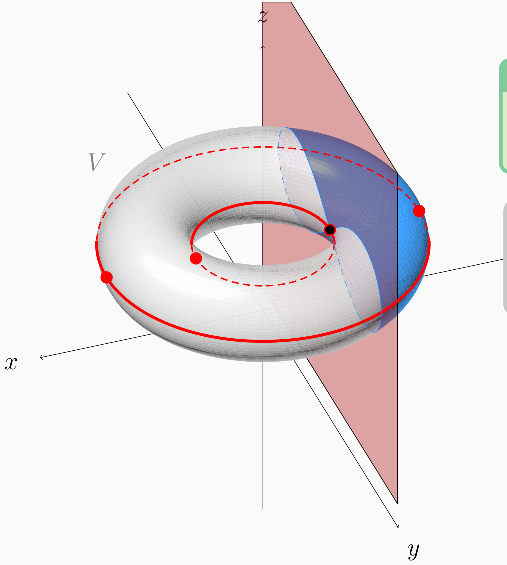
Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value



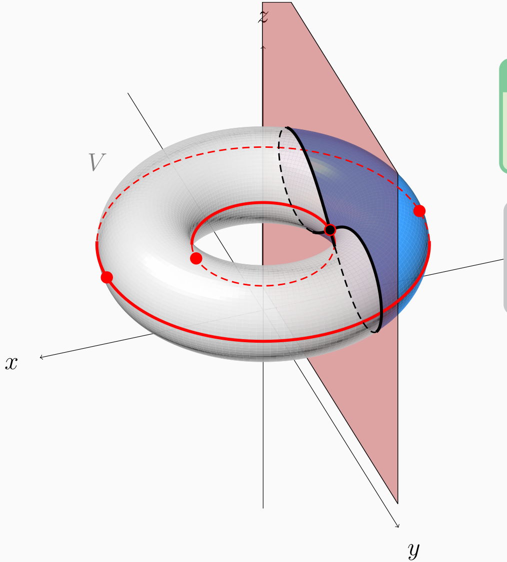
Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value



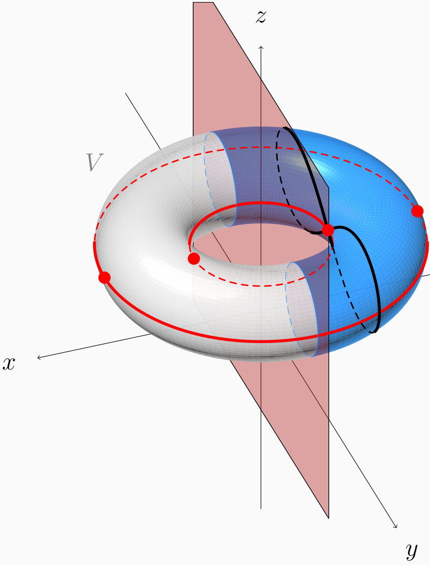
Roadmap property

$\forall C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value



Roadmap property

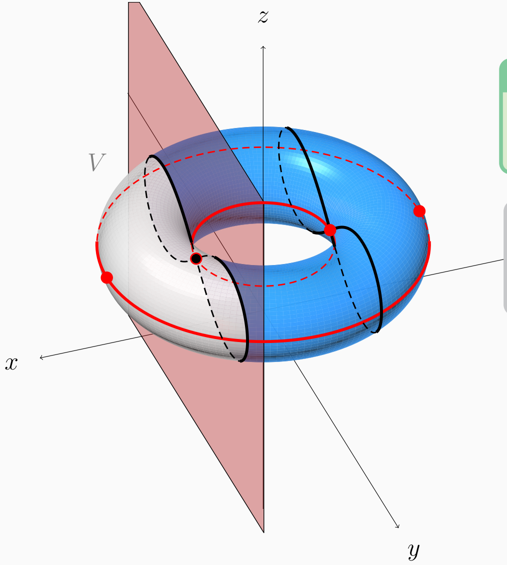
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

Canny's strategy



Roadmap property

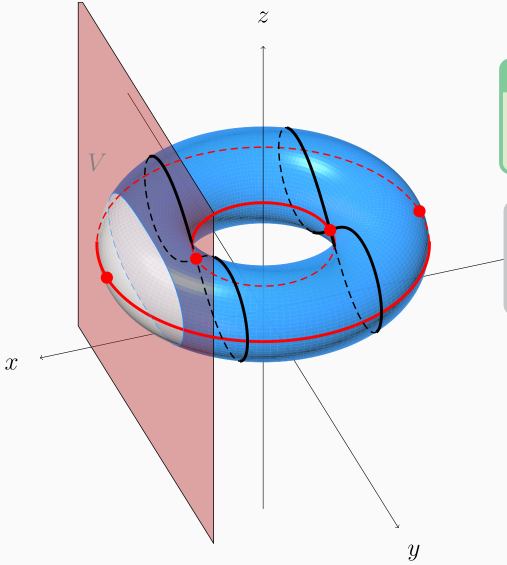
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

Canny's strategy



Roadmap property

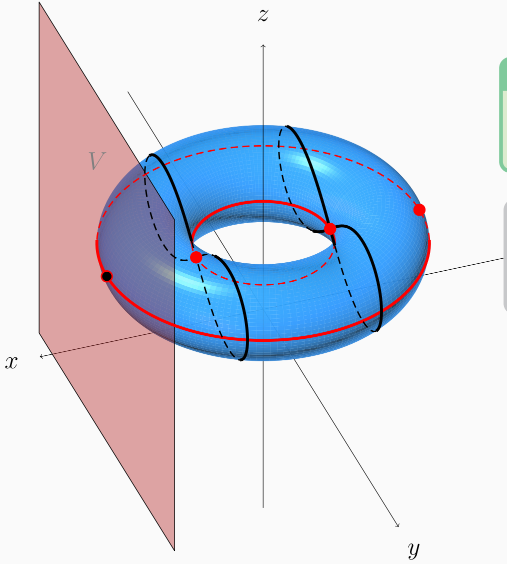
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

Canny's strategy



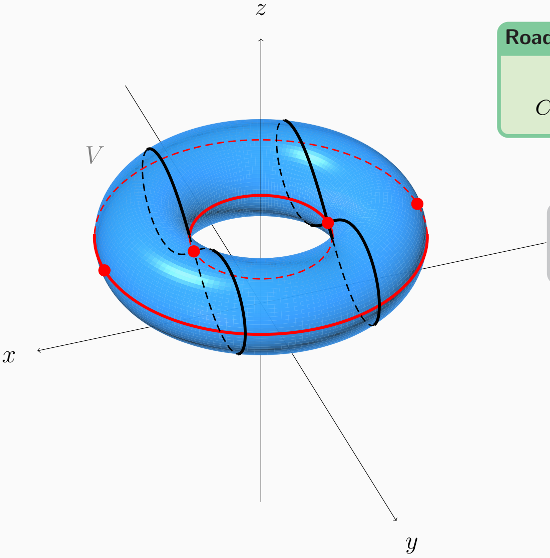
Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

Morse theory

"Scan" $\boldsymbol{W(\pi_2, V)}$ at the critical values of π_1

- We repair the connectivity failures with critical fibers
- We repeat the process at every critical value

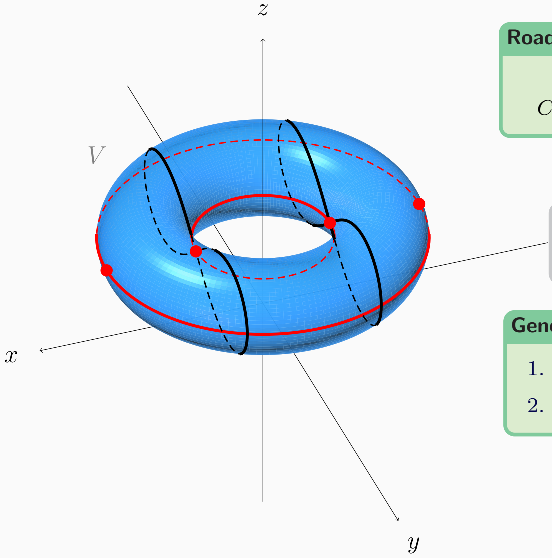


Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

$\boldsymbol{W(\pi_2, V)}$ **polar variety**
$\boldsymbol{F}$ **critical fibers**

Canny's strategy



Roadmap property

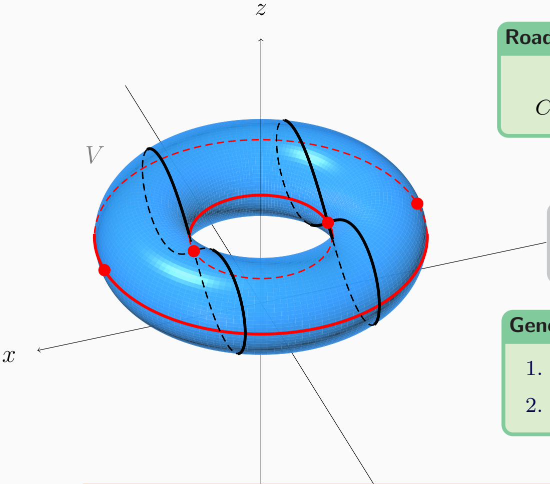
$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

$\boldsymbol{W(\pi_2, V)}$ **polar variety**
$\boldsymbol{F}$ **critical fibers**

Genericity assumptions

1. $\boldsymbol{W(\pi_2, V)}$ has dimension 1
2. $\boldsymbol{F}$ has dimension $\dim(V) - 1$

Canny's strategy



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

$W(\pi_2, V)$ polar variety
F critical fibers

Genericity assumptions

1. $W(\pi_2, V)$ has dimension 1
2. F has dimension $\dim(V) - 1$

Theorem [Canny, 1988]

If V is bounded, $W(\pi_2, V) \bigcup F$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of <u>dimension d</u> and defined by s polynomials of <u>degree $\leqslant D$</u>

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F$ has dimension $d-1$
and satisfies the Roadmap property.

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F$ has dimension $d-1$
and satisfies the Roadmap property.

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F$ has dimension $d-1$
and satisfies the Roadmap property.

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1} D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n \log^2 n)}$	Algebraic sets

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n\log^2 n)}$	Algebraic sets
[Safey El Din & Schost, 2017]	$(n^2D)^{\mathbf{6}n\log_2(d)+O(n)}$	Smooth, bounded algebraic sets

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of <u>dimension d</u> and defined by s polynomials of <u>degree $\leqslant D$</u>

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property

Author·s	Complexity	Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1} D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n \log^2 n)}$	Algebraic sets
[Safey El Din & Schost, 2017]	$(n^2 D)^{6n \log_2(d) + O(n)}$	Smooth, bounded algebraic sets
[P. & Safey El Din & Schost, 2024]	$(n^2 D)^{6n \log_2(d) + O(n)}$	Smooth, ~~bounded~~ algebraic sets

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property

Results based on a theorem in the **bounded** case		Assumptions
[Schwartz & Sharir, 1983]	$(sD)^{2^{O(n)}}$	
[Canny, 1993]	$(sD)^{O(n^2)}$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n\log^2 n)}$	Algebraic sets
[Safey El Din & Schost, 2017]	$(n^2D)^{6n\log_2(d)+O(n)}$	Smooth, bounded algebraic sets
[P. & Safey El Din & Schost, 2024]	$(n^2D)^{6n\log_2(d)+O(n)}$	Smooth, ~~bounded~~ algebraic sets

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property

Results based on a theorem in the **bounded** case		Assumptions
[Schwartz & Sharir, 1983]	$(sD$	
[Canny, 1993]	$(sD$	
[Basu & Pollack & Roy, 2000]	$s^{d+1} D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	Smooth, bounded algebraic sets
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n \log^2 n)}$	Algebraic sets
[Safey El Din & Schost, 2017]	$(n^2 D)^{6n \log_2(d) + O(n)}$	Smooth, bounded algebraic sets
[P. & Safey El Din & Schost, 2024]	$(n^2 D)^{6n \log_2(d) + O(n)}$	Smooth, ~~bounded~~ algebraic sets

Removing the boundedness
requires infinitesimals

On the complexity of computing roadmaps

$S \subset \mathbb{R}^n$ semi alg. set of <u>dimension d</u> and defined by s polynomials of <u>degree $\leqslant D$</u>

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property

Results based on a theorem in the **bounded** case

		Assumptions
[Schwartz & Sharir, 1983]	$(sD$	
[Canny, 1993]	$(sD$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	s
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n\log^2 n)}$	Algebraic sets
[Safey El Din & Schost, 2017]	$(n^2D)^{\mathbf{6}n\log_2(d)+O(n)}$	Smooth, bounded algebraic sets
[P. & Safey El Din & Schost, 2024]	$(n^2D)^{\mathbf{6}n\log_2(d)+O(n)}$	Smooth, ~~bounded~~ algebraic sets

Removing the boundedness requires infinitesimals

Not polynomial in the output size

$S \subset \mathbb{R}^n$ semi alg. set of dimension d and defined by s polynomials of degree $\leqslant D$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$ and satisfies the Roadmap property

Results based on a theorem in the **bounded** case

		Assumptions
[Schwartz & Sharir, 1983]	$(sI$	
[Canny, 1993]	$(sI$	
[Basu & Pollack & Roy, 2000]	$s^{d+1}D^{O(n^2)}$	
[Safey El Din & Schost, 2011]	$(nD)^{O(n\sqrt{n})}$	s
[Basu & Roy & Safey El Din & Schost, 2014]	$(nD)^{O(n\sqrt{n})}$	Algebraic sets
[Basu & Roy, 2014]	$(nD)^{O(n\log^2 n)}$	
[Safey El Din & Schost, 2017]	$(n^2D)^{6n\log_2(d)+}$	s
[P. & Safey El Din & Schost, 2024]	$(n^2D)^{6n\log_2(d)+}$	Smooth, bounded algebraic sets

Removing the boundedness requires infinitesimals

Not polynomial in the output size

Necessity of a new theorem in the **unbounded** case!

On the extension of Canny's result

Projection on 2 coordinates

$$\pi_2 : \begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^2 \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) & \mapsto & (\boldsymbol{x}_1, \boldsymbol{x}_2) \end{array}$$

- $W(\pi_2, V)$ polar variety
- $F_2 = \pi_1^{-1}(\pi_1(K)) \cap V$ critical fibers
- $K =$ critical points of π_1 on $W(\pi_2, V)$

Connectivity result [Canny, 1988]

If V is bounded, $W(\pi_2, V) \cup F_2$ has dimension $d - 1$
and satisfies the Roadmap property

On the extension of Canny's result

Projection on i coordinates

$$\pi_i : \begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^i \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) & \mapsto & (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i) \end{array}$$

- $W(\pi_i, V)$ polar variety
- $F_i = \pi_{i-1}^{-1}(\pi_{i-1}(K)) \cap V$ critical fibers
- $K =$ critical points of π_1 on $W(\pi_i, V)$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property

On the extension of Canny's result

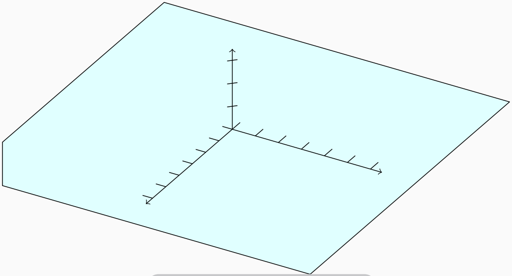
Projection on i coordinates

$$\pi_i : \begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^i \\ (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) & \mapsto & (\boldsymbol{x}_1, \dots, \boldsymbol{x}_i) \end{array}$$

- $W(\pi_i, V)$ polar variety
- $F_i = \pi_{i-1}^{-1}(\pi_{i-1}(K)) \cap V$ critical fibers
- $K =$ critical points of π_1 on $W(\pi_i, V)$

Connectivity result [Safey El Din & Schost, 2011]

If V is bounded, $W(\pi_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property



No critical points...

On the extension of Canny's result

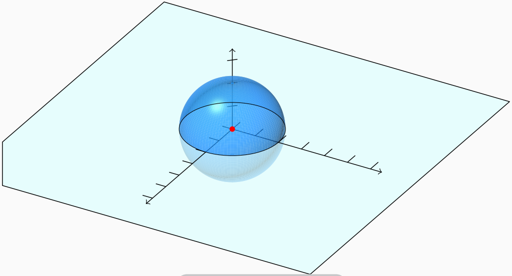
Non-negative ~~proper~~ polynomial map

$$\boldsymbol{\varphi}_i : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ \boldsymbol{x} & \mapsto & (\psi_1(\boldsymbol{x}), \ldots, \psi_i(\boldsymbol{x})) \end{array}$$

- $W(\boldsymbol{\varphi}_i, V)$ generalized polar variety
- $F_i = \boldsymbol{\varphi}_{i-1}^{-1}(\boldsymbol{\varphi}_{i-1}(K)) \cap V$ critical fibers.
- K = critical points of $\boldsymbol{\varphi}_1$ on $W(\boldsymbol{\varphi}_i, V)$

Connectivity result [P. & Safey El Din & Schost, 2024] NEW!

~~If V is bounded,~~ $W(\boldsymbol{\varphi}_i, V) \cup F_i$ has dimension $\max(i-1, d-i+1)$
and satisfies the Roadmap property



Critical point!

🛠

⇝ Sard's lemma

⇝ Thom's isotopy lemma

⇝ Puiseux series

How to use it?

Assumptions to satisfy in the new result

(R) $\operatorname{sing}(V)$ is finite

(P) φ_1 is a proper map bounded from below

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i

(W) $\dim W_i = i - 1$ and $\operatorname{sing}(W_i) \subset \operatorname{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\operatorname{sing}(F_i)$ is finite

How to use it?

Assumptions to satisfy in the new result

(R) $\text{sing}(V)$ is finite ✓

(P) φ_1 is a proper map bounded from below

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i

(W) $\dim W_i = i - 1$ and $\text{sing}(W_i) \subset \text{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\text{sing}(F_i)$ is finite

🛠

Assumption on the input

How to use it?

Assumptions to satisfy in the new result

(R) $\mathrm{sing}(V)$ is finite ✓

(P) φ_1 is a proper map bounded from below ✓

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i

(W) $\dim W_i = i - 1$ and $\mathrm{sing}(W_i) \subset \mathrm{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\mathrm{sing}(F_i)$ is finite

🛠

By construction of φ

A successful candidate

Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \, , \, \boldsymbol{b}_2^\top \vec{\boldsymbol{x}} \, , \, \ldots \, , \, \boldsymbol{b}_n^\top \vec{\boldsymbol{x}} \right) \quad \text{where} \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^n$$

It satisfies the assumptions! NEW!

How to use it?

Assumptions to satisfy in the new result

(R) $\mathrm{sing}(V)$ is finite ✓

(P) φ_1 is a proper map bounded from below ✓

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i ✓

(W) $\dim W_i = i - 1$ and $\mathrm{sing}(W_i) \subset \mathrm{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\mathrm{sing}(F_i)$ is finite

Generalization of Noether position from [Safey El Din & Schost, 2003]

A successful candidate

Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \,,\, \boldsymbol{b}_2^{\mathsf{T}} \vec{\boldsymbol{x}} \,,\, \dots \,,\, \boldsymbol{b}_n^{\mathsf{T}} \vec{\boldsymbol{x}} \right) \quad \text{where} \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^n$$

It satisfies the assumptions! NEW!

How to use it?

Assumptions to satisfy in the new result

(R) $\mathrm{sing}(V)$ is finite ✓

(P) φ_1 is a proper map bounded from below ✓

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i ✓

(W) $\dim W_i = i - 1$ and $\mathrm{sing}(W_i) \subset \mathrm{sing}(V)$ ✓

(F) $\dim F_i = n - d + 1$ and $\mathrm{sing}(F_i)$ is finite

Jacobian criterion

$\oplus$

Thom's transversality theorem

A successful candidate

Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \,,\, \boldsymbol{b}_2^{\mathsf{T}} \vec{\boldsymbol{x}} \,,\, \ldots \,,\, \boldsymbol{b}_n^{\mathsf{T}} \vec{\boldsymbol{x}} \right) \quad \text{where} \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^n$$

It satisfies the assumptions! NEW!

How to use it?

Assumptions to satisfy in the new result

(R) $\operatorname{sing}(V)$ is finite

(P) φ_1 is a proper map bounded from below

For all $1 \leqslant i \leqslant \dim(V)/2$,

(N) φ_{i-1} has finite fibers on W_i

(W) $\dim W_i = i - 1$ and $\operatorname{sing}(W_i) \subset \operatorname{sing}(V)$

(F) $\dim F_i = n - d + 1$ and $\operatorname{sing}(F_i)$ is finite

Jacobian criterion

$\oplus$

Noether position

A successful candidate

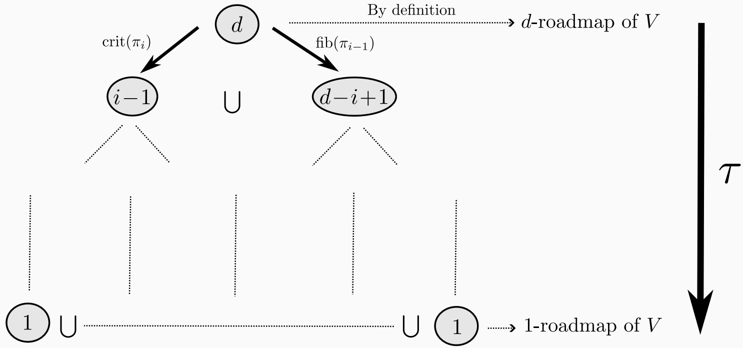
Choose *generic* $(\boldsymbol{a}, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n) \in \mathbb{R}^{n^2}$ and:

$$\boldsymbol{\varphi} = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \,,\, \boldsymbol{b}_2^{\mathsf{T}} \vec{\boldsymbol{x}} \,,\, \ldots \,,\, \boldsymbol{b}_n^{\mathsf{T}} \vec{\boldsymbol{x}} \right) \quad \text{where} \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^n$$

It satisfies the assumptions! NEW!

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

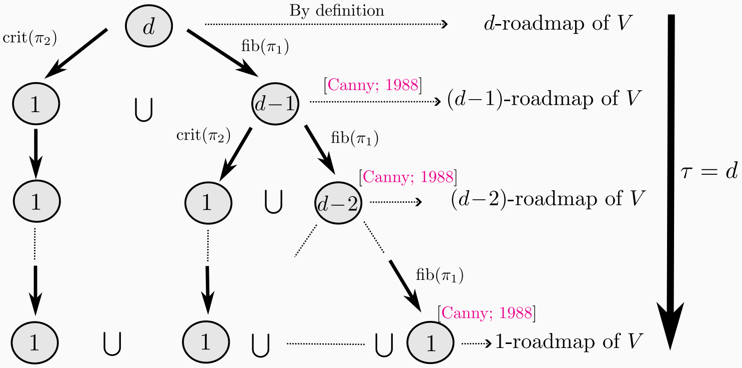


Depth of recursion tree : τ

$\Rightarrow$ complexity: $(nD)^{O(n\tau)}$

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

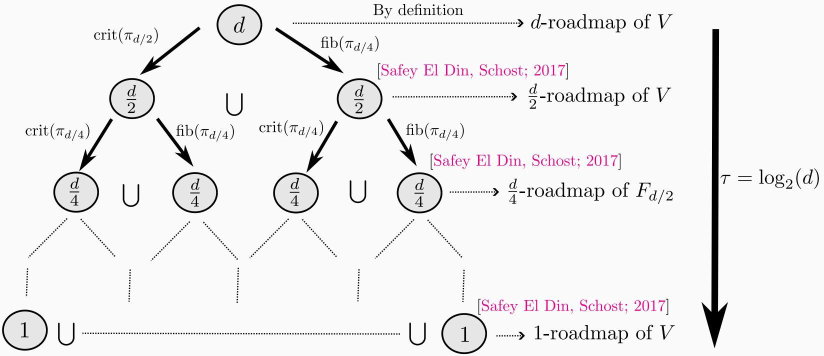


Depth of recursion tree : d

$\Rightarrow$ complexity: $(nD)^{O(nd)}$

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

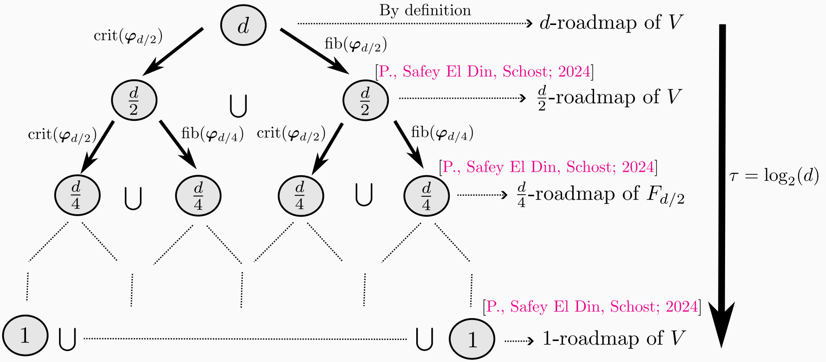


Depth of recursion tree : $\log_2(d)$

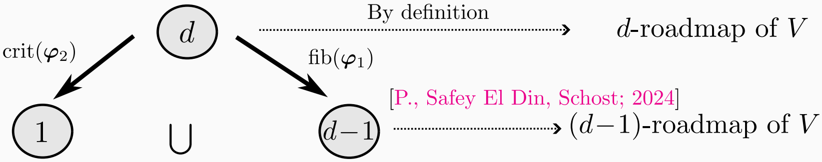
$\Rightarrow$ complexity: $(nD)^{O(n \log_2(d))}$

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

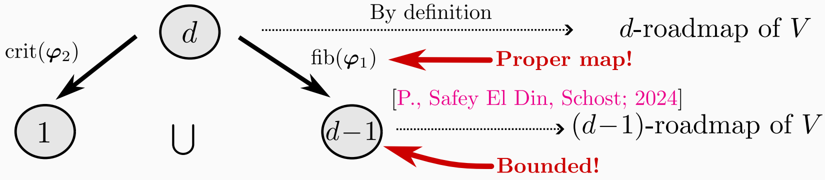


Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



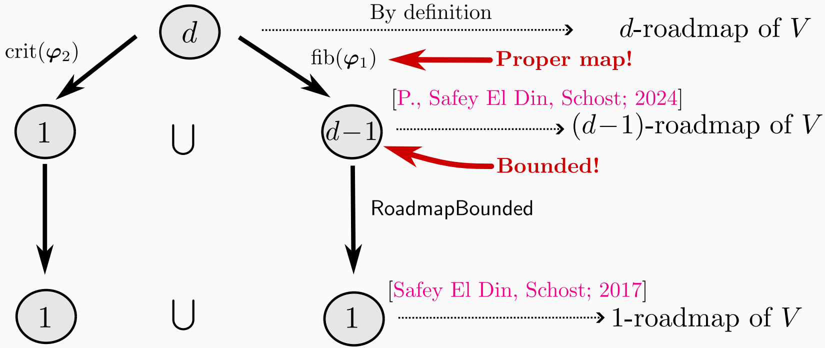
An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



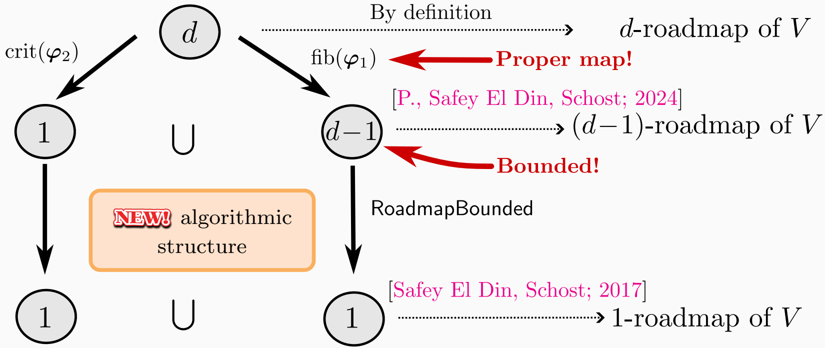
An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

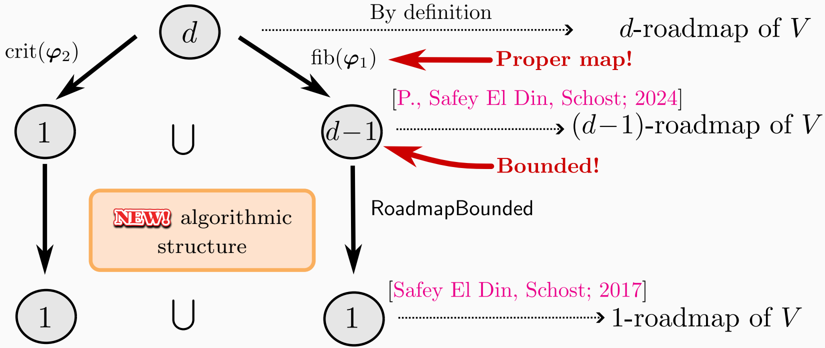


Quantitative estimate

	Output size	Complexity
RoadmapBounded(fib(φ_1)) Compute crit(φ_2) & fib(φ_1)		
Overall		

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

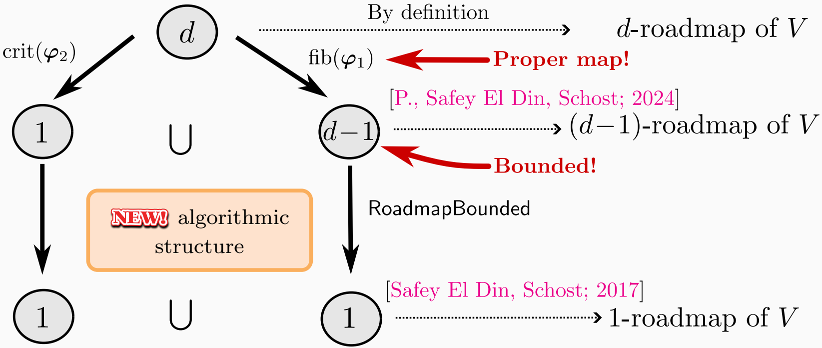


Quantitative estimate

	Output size	Complexity
RoadmapBounded$(\mathrm{fib}(\boldsymbol{\varphi}_1))$ Compute $\mathrm{crit}(\boldsymbol{\varphi}_2)$ & $\mathrm{fib}(\boldsymbol{\varphi}_1)$	$(n^2D)^{4n\log_2 d + O(n)}$	$(n^2D)^{6n\log_2 d + O(n)}$
Overall		

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

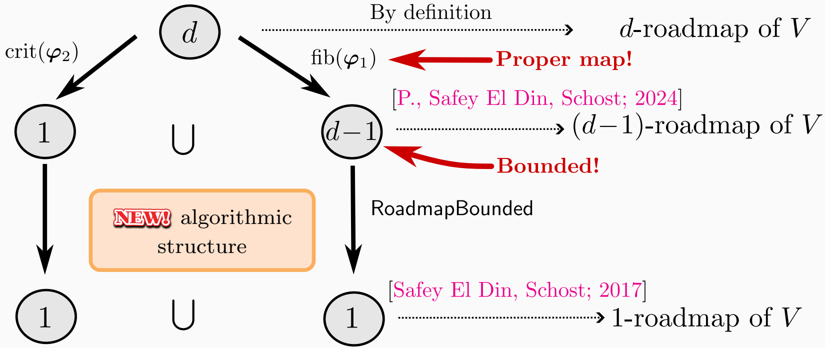


Quantitative estimate

	Output size	Complexity
RoadmapBounded(fib(φ_1))	$(n^2 D)^{4n \log_2 d + O(n)}$	$(n^2 D)^{6n \log_2 d + O(n)}$
Compute crit(φ_2) & fib(φ_1)	$(nD)^{O(n)}$	$(nD)^{O(n)}$
Overall		

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d

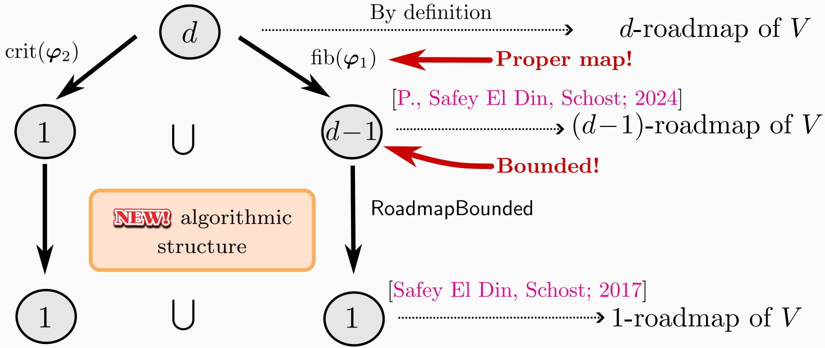


Quantitative estimate

	Output size	Complexity
RoadmapBounded($\mathrm{fib}(\varphi_1)$)	$(n^2D)^{4n\log_2 d+O(n)}$	$(n^2D)^{6n\log_2 d+O(n)}$
Compute $\mathrm{crit}(\varphi_2)$ & $\mathrm{fib}(\varphi_1)$	$(nD)^{O(n)}$	$(nD)^{O(n)}$
Overall	$(n^2D)^{4n\log_2 d+O(n)}$	$\boxed{(n^2D)^{6n\log_2 d+O(n)}}$

An algorithm for unbounded algebraic set

Consider an algebraic set $V \subset \mathbb{C}^n$ with dimension d



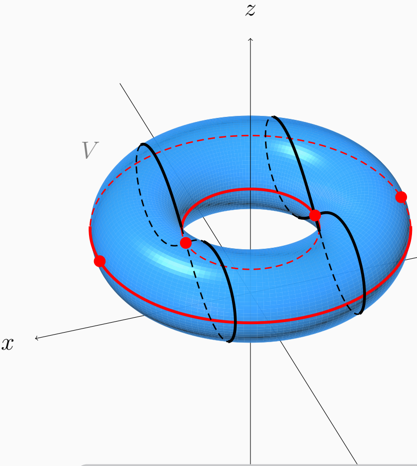
Quantitative estimate

	Output size	Complexity
RoadmapBounded($\mathrm{fib}(\varphi_1)$)	$(n^2 D)^{4n \log_2 d + O(n)}$	$(n^2 D)^{6n \log_2 d + O(n)}$
Compute $\mathrm{crit}(\varphi_2)$ & $\mathrm{fib}(\varphi_1)$	$(nD)^{O(n)}$	$(nD)^{O(n)}$
Overall	$(n^2 D)^{4n \log_2 d + O(n)}$	$\boxed{(n^2 D)^{6n \log_2 d + O(n)}}$

Analysis of the kinematic singularities of a PUMA robot

with J.Capco, M.Safey El Din and P.Wenger

Canny's strategy



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

$\boldsymbol{W(\pi_2, V)}$ **polar variety**
$\boldsymbol{F}$ **critical fibers**

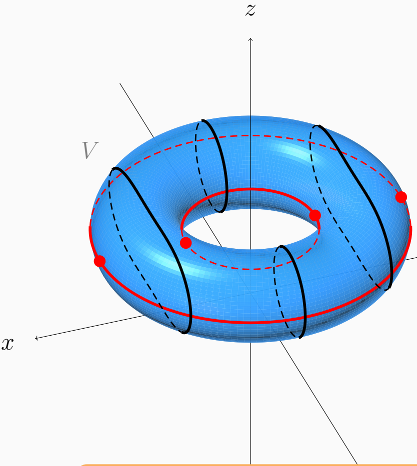
Genericity assumptions

1. $\boldsymbol{W(\pi_2, V)}$ has dimension 1
2. $\boldsymbol{F}$ has dimension $\dim(V) - 1$

Theorem [Canny, 1988]

If V is bounded, $\boldsymbol{W(\pi_2, V)} \bigcup \boldsymbol{F}$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property

Canny's strategy



Roadmap property

$\forall\, C$ connected component,
$C \cap \mathscr{R}$ is non-empty and connected

$W(\pi_2, V)$ **polar variety**
F **regular fibers**

Genericity assumptions

1. $W(\pi_2, V)$ has dimension 1
2. F has dimension $\dim(V) - 1$

Theorem [Mezzaroba & Safey El Din, 2006]

If V is bounded, $W(\pi_2, V) \bigcup F$ has dimension $\dim(V) - 1$
and satisfies the Roadmap property

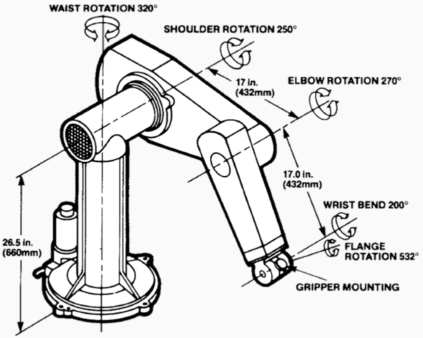
Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix}
(v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\
0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\
0 & 1 & 0 & 0 & 0 & 2a_3 \\
0 & 0 & 1 & 0 & 0 & 0 \\
v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\
(v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1)
\end{bmatrix}$$

`https://msolve.lip6.fr`

⇝ Multivariate system solving
⇝ Real roots isolation



A PUMA 560 [Unimation, 1984]

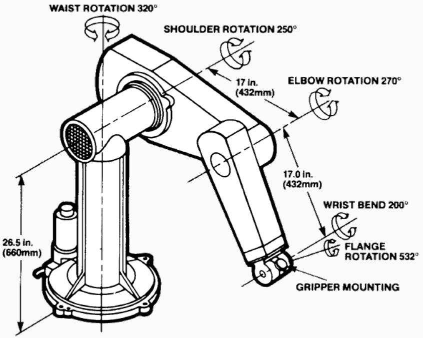
Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix}
(v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\
0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\
0 & 1 & 0 & 0 & 0 & 2a_3 \\
0 & 0 & 1 & 0 & 0 & 0 \\
v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\
(v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1)
\end{bmatrix}$$

`https://msolve.lip6.fr`

⇝ Multivariate system solving
⇝ Real roots isolation



A PUMA 560 [Unimation, 1984]

First step

Max. deg without splitting: **1858**

Locus	Degrees	ℝ-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46

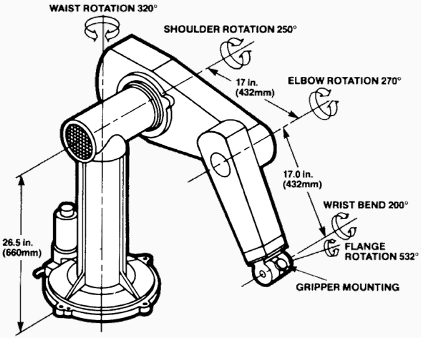
Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix} (v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\ 0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\ 0 & 1 & 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\ (v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1) \end{bmatrix}$$

`https://msolve.lip6.fr`

- ⇝ Multivariate system solving
- ⇝ Real roots isolation



A PUMA 560 [Unimation, 1984]

First step

Max. deg without splitting: **1858**

Locus	Degrees	$\mathbb{R}$-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46

Recursive step over 95 fibers

Data are for one fiber

Locus	Degrees	$\mathbb{R}$-roots	Total time
Critical points	38	14	6.4 min
Critical curves	21	∞	9.6 min

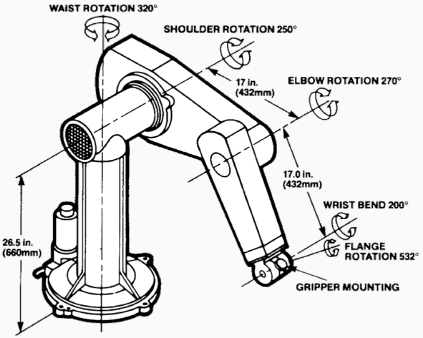
Roadmap computation for robotics

Matrix M associated to a PUMA-type robot with a non-zero offset in the wrist

$$\begin{bmatrix} (v_3+v_2)(1-v_2v_3) & 0 & A(\boldsymbol{v}) & d_3A(\boldsymbol{v}) & a_2(v_3^2+1)(v_2^2-1)-a_3A(\boldsymbol{v}) & 2d_3(v_3+v_2)(v_2v_3-1) \\ 0 & v_3^2+1 & 0 & 2a_2v_3 & 0 & (a_3-a_2)v_3^2+a_2+2a_3 \\ 0 & 1 & 0 & 0 & 0 & 2a_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 1-v_4^2 & 0 & d_4(1-v_4^2) & -2d_4v_4 & 0 \\ (v_4^2-1)v_5 & 4v_4v_5 & (1-v_5^2)(v_4^2+1) & (1-v_5^2)(v_4^2-1)d_5+4d_4v_4v_5 & 2d_5v_4(1-v_5^2)+2d_4v_5(1-v_4^2) & -2d_5v_5(v_4^2+1) \end{bmatrix}$$

`https://msolve.lip6.fr`

⇝ Multivariate system solving
⇝ Real roots isolation



A PUMA 560 [Unimation, 1984]

First step

Max. deg without splitting: **1858**

Locus	Degrees	$\mathbb{R}$-roots	Tot. time
Critical points	400 & 934	96 & 182	9.7 min
Critical curves	182 & 220	∞	3h46

Recursive step over 95 fibers

Data are for one fiber

Locus	Degrees	$\mathbb{R}$-roots	Total time
Critical points	38	14	6.4 min
Critical curves	21	∞	9.6 min

Roadmap computation NEW!

Output degree: **4847**
Time: **4h10** (`msolve`)

Perspectives

Algorithms

Roadmap algorithms:

- Adapt the algorithms to structured systems: quadratic case (J.A.K.Elliott, M.Safey El Din, É.Schost)
- Generalize the connectivity result to semi-algebraic sets
- Design optimal roadmap algorithms with complexity exponential in $O(n)$

Connectivity of s.a. curves:

- Adapt to algebraic curves given as union (A.Poteaux)
- Generalize to semi-algebraic curves

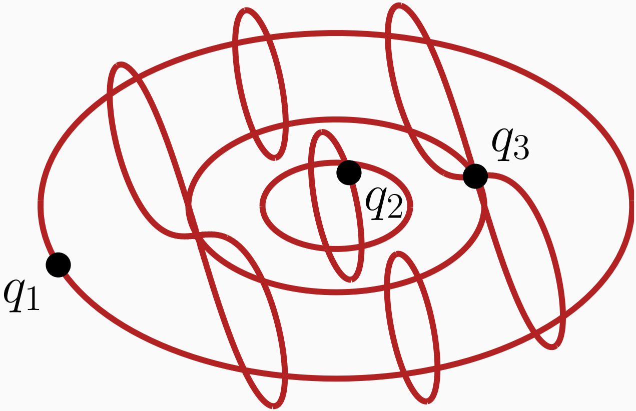
Applications

- Analyze challenging class of robots (D.Salunkhe, P.Wenger)
- Obtain practical version of modern roadmap algorithms

Software

- Curves: subresultant/GCD computations deg ~ 100 (now) $\to \sim 200$ (target)
- Computational real algebraic geometry library as part of AlgebraicSolving.jl (C.Eder, R.Mohr, R.P., M.Safey El Din)
- Implement a ready-to-use toolbox for roboticians

Union of curves



Reduce data size