Braiding Vineyards

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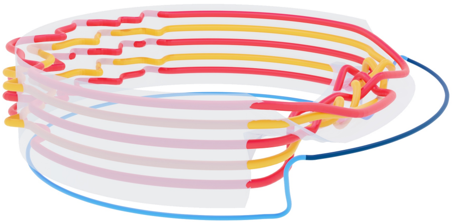

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Knot theory

Persistence

Main results

Main Theorem

Given a knot, there exists a manifold $\mathcal{M} \subset \mathbb{R}^d$ and a closed curve $\gamma \subset \mathbb{R}^d$ such that identifying the ends of the l-vineyard of $d(\cdot, \gamma(t))|_{\mathcal{M}}$ will yield knot, which contains the given knot (there are some spurious components that will have to be removed).

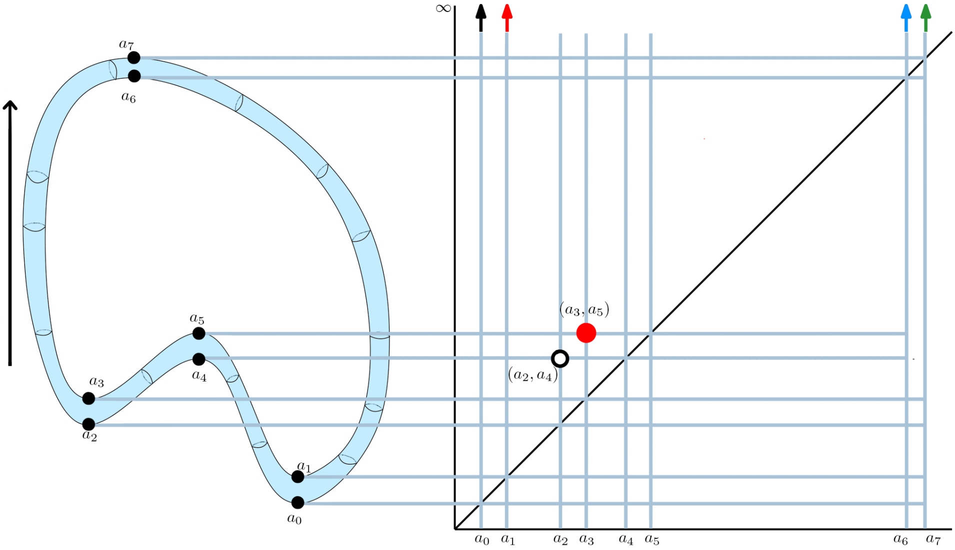
Main Corollary

Monodromy of any order k can be created in the l-vineyard of the radial distance function restricted to a manifold, $\mathcal{M} \subset \mathbb{R}^d$.

Preliminaries

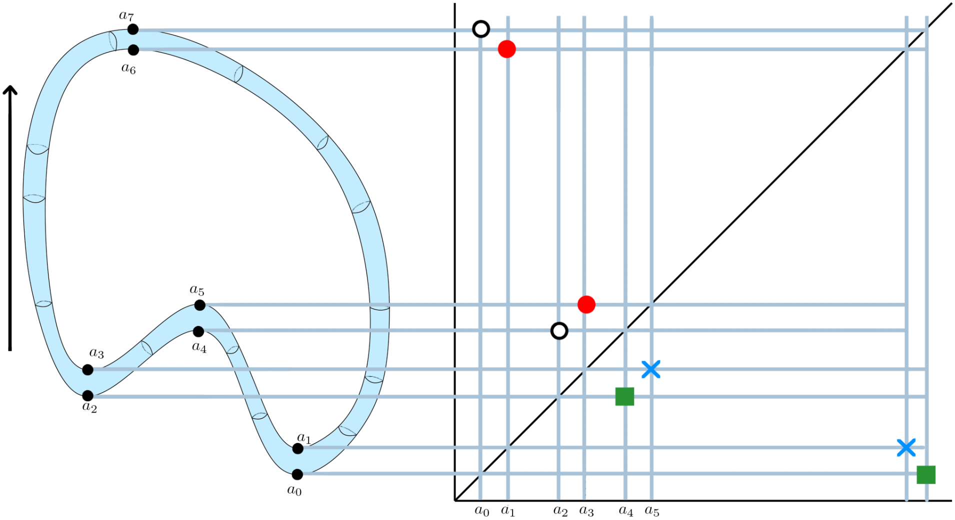
Persistence

The example $\mathbb{S}^1 \times \mathbb{S}^l$



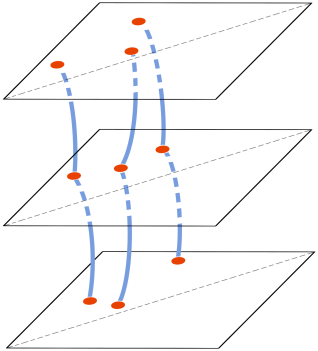
Extended Persistence

The example $\mathbb{S}^1 \times \mathbb{S}^l$



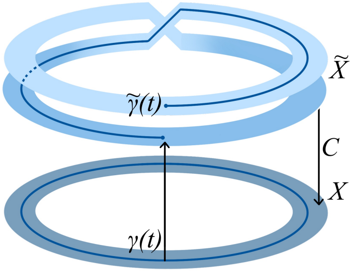
Vineyards

- Persistence is stable with respect to small perturbations to the filtration [Cohen-Steiner et al., 2005]
- Therefore we can study persistence arising from a 1-parameter family of functions, $f_t : \mathcal{M} \to \mathbb{R}$ for $t \in [0, T]$
- We can "stack" the persistence diagrams according to t and trace points in diagrams as vines through the Vineyard of diagrams [Cohen-Steiner et al., 2006]



Monodromy

- Monodromy is the effect where a loop in a base space may not lift to a loop in a covering or fibre bundle
- Let $C : \tilde{X} \to X$ be a covering map
- For a curve $\gamma : [0, 2\pi] \to X$, we write $\tilde{\gamma}$ for (one of) its lift(s)
- If γ is a loop and $\tilde{\gamma}(0) \neq \tilde{\gamma}(2\pi)$, then we say that γ exhibits monodromy (at the starting point)
- We say γ exhibits monodromy of order k if k is the smallest positive integer such that $\widetilde{\gamma^k}(0) = \widetilde{\gamma^k}(2\pi k)$, for a concatenated loop $\gamma^k = \underbrace{\gamma \circ \cdots \circ \gamma}_{k}$



Definitions and previous work

Previous work

- Monodromy in TDA was observed in the context of multiparameter persistence by Cerri, Ethier, and Frosini (2013). Recent follow-up in Scaramuccia & Mortain (2025).
- Independently it was found in (single parameter) persistence by Arya, Giunti, Hickok, Kanari, McGuire, and Turner (2024).
  Their example was constructed with the persistent homology transform.
- Onus, Nina, and Turkes (2024) generalized persistent homology transform to arbitrary dimension, i.e. instead of distance in a single direction, distance to an affine flat.

Radial Distance Function

We consider an extreme version and consider the distance to a point:

- Let $\mathcal{M} \subset \mathbb{R}^d$. We define the radial distance function to be $d(\cdot, x)|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ that is the Euclidean function from x restricted to $\mathcal{M} \subset \mathbb{R}^d$.

Previous work on the distance function

- Bruce, Giblin, and Gibson (1985) studied the distance function $d(\cdot, x)|_{\mathcal{M}}$ from a singularity theory perspective. This yields the symmetry set: The set of all points in the ambient space such that a sphere centred at this point is tangent to $\mathcal{M}$ in multiple places.
- Yomdin (1981) and Mather (1983) studied the singularities of the distance to the manifold $d(\cdot, \mathcal{M})$, that is the singular structure of the medial axis. The medial axis (Erdős (1945), Federer (1959)) is a subset of the symmetry set and consist of all point in the ambient space for which there is no unique closest point on the manifold.

From the Radial Distance Function to Vineyards

- Let $d(\cdot, x)_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ be the radial distance function from x restricted to $\mathcal{M} \subset \mathbb{R}^d$
- Setting $x = \gamma(t)$ for $\gamma : [0, 2\pi] \to \mathbb{R}^d$, we obtain a family of filtrations, $d(\cdot, \gamma(t))_{\mathcal{M}}$
- We call γ the observation loop
- The closed vineyard map

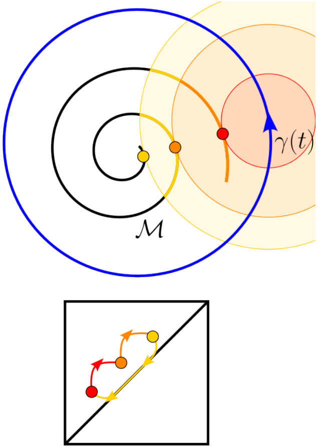
$$\begin{aligned} CV_{\mathcal{M}} : \mathbb{S}^1 &\to \mathbb{S}^1 \times \mathrm{Dgm} \\ t &\to (t, \mathrm{Dgm}(d(\cdot, \gamma(t))_{\mathcal{M}})) \end{aligned}$$

is a covering map of $\gamma \simeq \mathbb{S}^1$

Monodromy in Vineyards

Arya et al.'s example (adjusted to our setting)

- The vines induce a map from $\mathrm{Dgm}_l(d(\cdot, \gamma(0))_{\mathcal{M}})$ to itself, which permutes the points in the persistence diagram
- A vineyard demonstrates monodromy of order k if k is the smallest integer $k > 0$ such that applying this permutation k times yields the identity permutation
- The spiral construction exhibits monodromy of order 3



$\mathrm{Dgm}_0(d(\cdot, \gamma(t))_{\mathcal{M}})$

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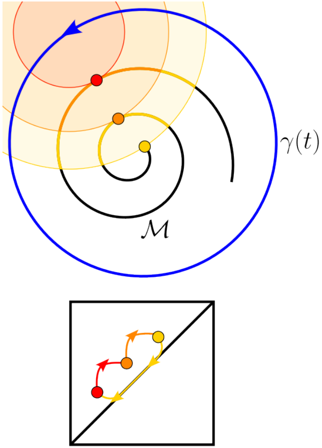

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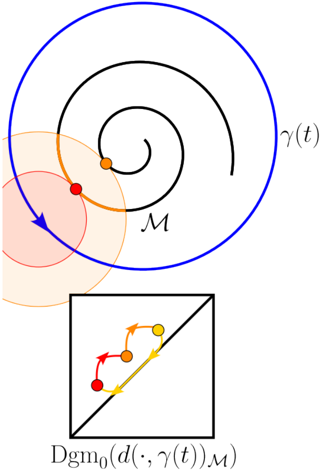
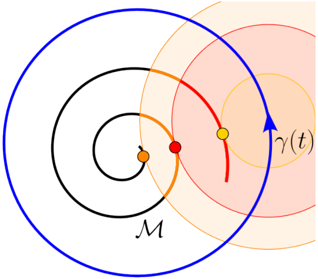
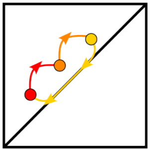


Monodromy in Vineyards

Arya et al.'s example (adjusted to our setting)

- The vines induce a map from $\mathrm{Dgm}_l(d(\cdot, \gamma(0))_{\mathcal{M}})$ to itself, which permutes the points in the persistence diagram
- A vineyard demonstrates monodromy of order k if k is the smallest integer $k > 0$ such that applying this permutation k times yields the identity permutation (we connect along the diagonal in an 'optimal' way)
- The spiral construction exhibits monodromy of order 3

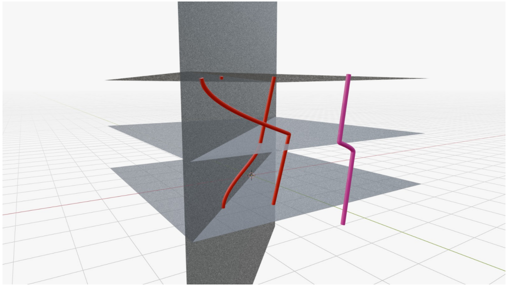




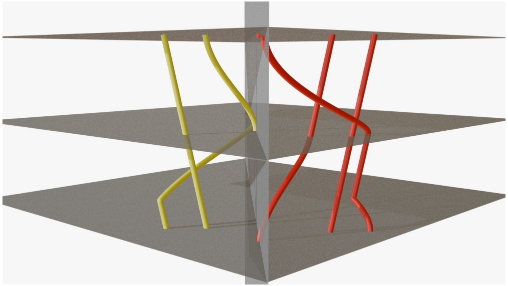
$\mathrm{Dgm}_0(d(\cdot, \gamma(t))_{\mathcal{M}})$

Monodromy in Vineyards

(Non-extended) Persistence



Extended Persistence



Arya et al. and an open question

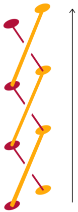
- [Arya et al., 2024] investigated monodromy in 0-vineyards of objects embedded in $\mathbb{R}^2$ through all height filtrations
- Open question: demonstrate monodromy in higher dimensional Vineyards

As we'll see the answer to this question is a corollary of (the proof of) the main result.

Knots and braids

Braids and knots

- A braid on m strands is the equivalence class of the disjoint union of m intervals, $B_i \colon I \to D^2 \times I$, monotonically increasing with respect to I, such that the end points are a permutation of the start points, under ambient braid isotopy



- By identifying the ends of the interval and mapping to the canonical solid torus in $\mathbb{R}^3$, we obtain a closed braid

Alexander's Theorem [Alexander, 1923]

Every knot or link is equivalent to a closed braid

We have a polynomial bound on the number of strands and crossings in the braid

Main result and proof

Main theorem

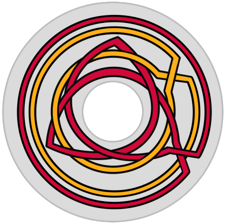
Given a braid B, there exists an $\mathcal{M} \subset \mathbb{R}^d$ and a closed curve $\gamma \subset \mathbb{R}^d$ such that identifying the ends of the ℓ-vineyard of $d(\cdot, \gamma(t))_{\mathcal{M}}$ will yield a braid B', which is equivalent to B after removing some spurious unbraided connected components.

- The proof is constructive, the choices of not only $\mathcal{M}$ and γ are critical
- Different γ can yield different braids (incorrect crossings) in the vineyard

Construction

Step 1:

- A given knot corresponds to a closed braid $B \subset \mathbb{R}^3$, by Alexander
- Assume that the closed braid B has K components and s strands
- Represent B as an almost annular closed braid and strands follow fixed radii
- Introduce an additional twist per connected component of the closed braid, this introduces an additional $\mathcal{O}(s \cdot K)$ crossings and total $n = s + K$ strands



Step 2:

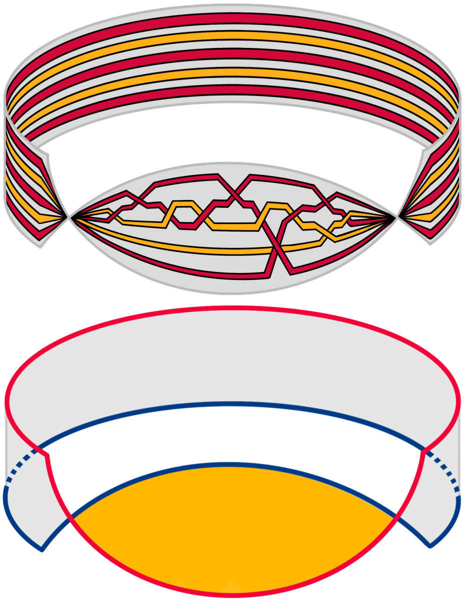
- Further deform B so that all crossings occur at regular intervals within an angular fraction of the total annulus' period

Step 3:

- Twist the annulus, outside the crossing region, so that it would intersect the original annulus orthogonally

Step 4:

- Set the observation loop γ to follow the the twisted annulus at some fixed distance from the core of the annulus



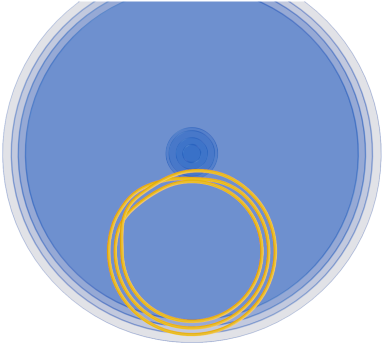
Lemma

At this point of the construction we have a closed braid near an annulus. Because the annulus can be very thin (small difference between inner and outer radius), it is close to a circle.

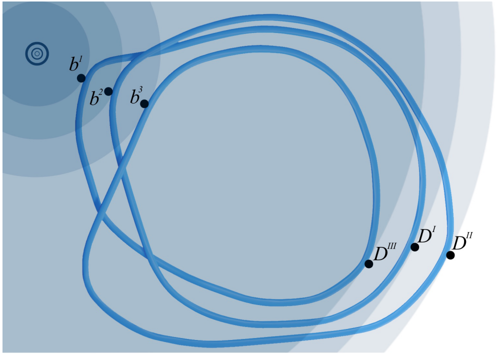
Lemma

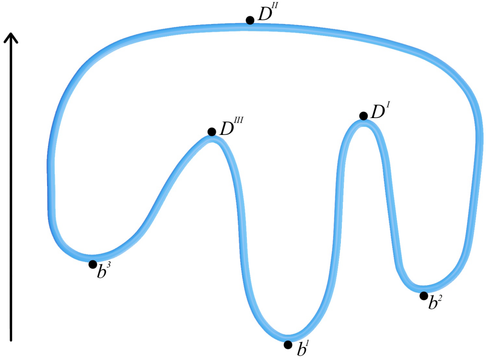
If a circular braid is embedded close to a circle, which is in turn close to the observation loop γ, then we have that:

- The morse critical points of $d(\cdot, \gamma(t))|_{\mathcal{M}}$ for $\gamma(t)$ near the circle split into two clusters, one near $\gamma(t)$ and one opposite (on the circle)
- For H_0 (in extended persistence) the all the critical points near $\gamma(t)$ correspond to births and all the points far away correspond to deaths. (For H_ℓ things are more complicated.)



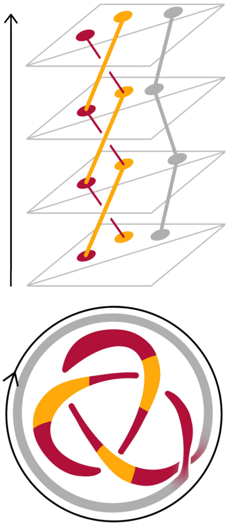
Equivalent embeddings





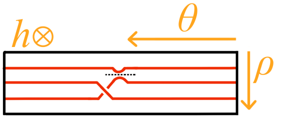
Observations

- The elder rule of persistence says that the first birth and final death are paired in each component of B
- The additional crossing/strand was introduced to account for this
- For each component of B there will be an unlinked strand which will become a loose circle in the closed braid vineyard (in extended persistence, in regular persistence this lies at infinity)

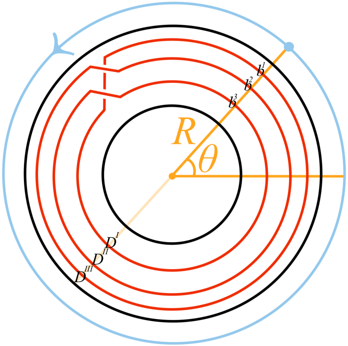


Observations

- The remaining vines have annular coordinates $(\theta, \rho, h) = (\theta, R - b^j, D^{J'} - 2R)$
- Then apart from around the crossings each vine has a distinct ρ value in which case the h value is not important
- We need to tinker with the death values at the crossings...

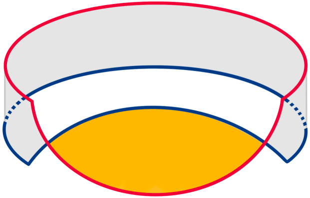


$$(\theta,\ \underbrace{R\text{-}b^j}_{\rho},\ \underbrace{D^{J'}\text{-}2R}_{h})$$



Why we needed the twist...

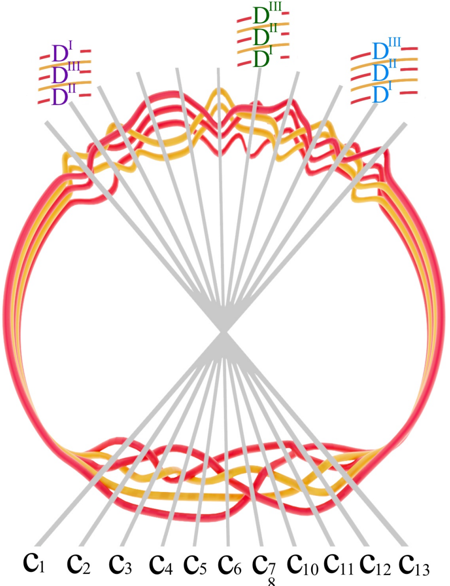
The twist makes sure that we can deform the braid, in such a way that it only influences either the birth or the death times.



Construction

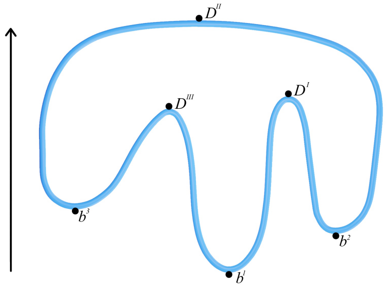
Step 5:

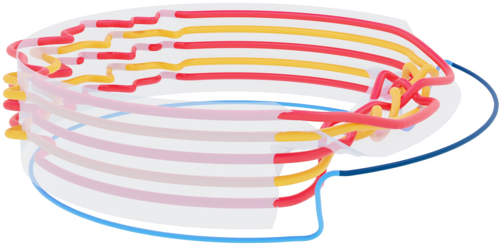
- Recall that all crossings are equally spaced and their death values occur at the opposite point in the braid
- Therefore we can "push" and "pull" the strands corresponding to the crossing in the opposite region the recreate the correct over/under crossing of vines in the vineyard



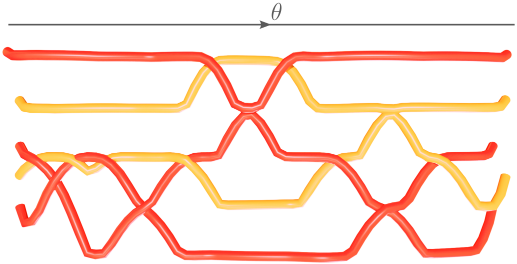
How to push/pull

The way you want to push/pull can be 'read off' from the equivalent embedding. In the example: If D^I is higher than D^{III} then the cycle born at b^2 passes over the cycle born at b^3 in the vineyard if we interchange the order of birth of b^2 and b^3.





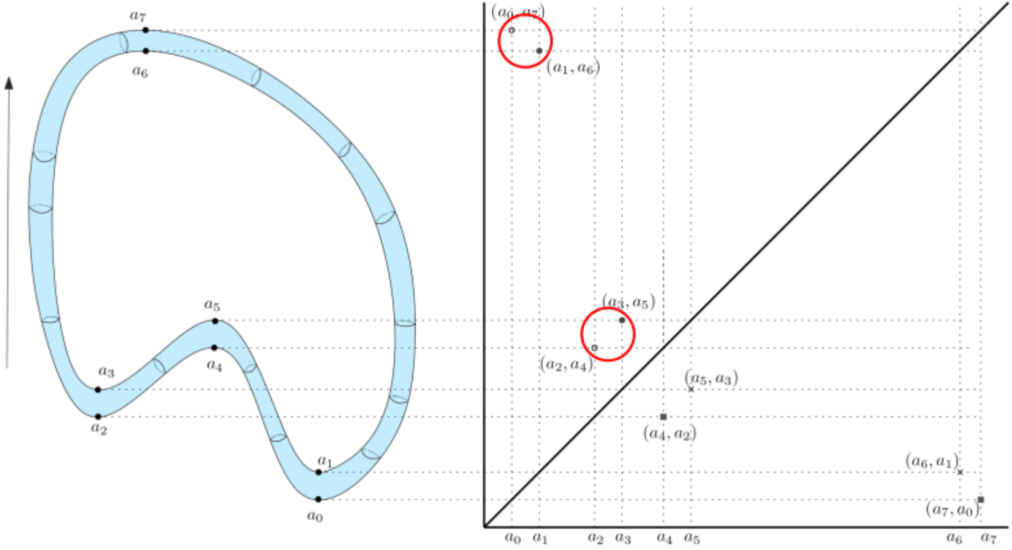
The construction



Vineyard corresponding to the dark blue region of the observation curve

Extending to l-persistence in $\mathbb{R}^d$

- Let $\mathcal{M}$ be the $(l-1)$-dimensional α-offset of $B \times 0$ in $\mathbb{R}^3 \bigoplus \mathbb{R}^{l-1} \simeq \mathbb{R}^d$
- Vines in each l-vineyard closely follow the vines of the 0-vineyard



a_7
a_6
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(a_1, a_6)
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(a_5, a_3)
(a_4, a_2)
(a_6, a_1)
(a_7, a_0)
a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7

Conclusion

- Vineyards of the radial distance function can be as topologically rich as possible
- This reinforces the work of [Onus et al., 2024]
  - The radial distance transform is topologically richer than the standard persistent homology transform
- It would be nice to define distances between closed vineyards that completely respect topology
  - This will necessarily involve knot recognition which is very difficult!
  - Are there combinations of existing distance and link invariants that can be computed efficiently?
- Do non trivial knots and links appear in real periodic data?

Questions?